

ALMOST SURE LOCAL WELLPOSEDNESS OF ENERGY CRITICAL FRACTIONAL SCHRÖDINGER EQUATIONS WITH HARTREE NONLINEARITY

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ABSTRACT. We consider a Cauchy problem of energy-critical fractional Schrödinger equation with Hartree nonlinearity below the energy space. Using a method of randomization of functions on \mathbb{R}^d associated with the Wiener decomposition, introduced by Á. Bényi, T. Oh, and O. Pocovnicu [3, 4], we prove that the Cauchy problem is almost surely locally well-posed. Our result includes Hartree Schrödinger equation ($\alpha = 2$).

1. INTRODUCTION

In this paper we consider the following Cauchy problem of the fractional nonlinear Schrödinger equations:

$$(1.1) \quad \begin{cases} i\partial_t u = |\nabla|^\alpha u + F(u) & \text{in } \mathbb{R}^{1+d}, \\ u(x, 0) = \phi(x) \in H^s & \text{in } \mathbb{R}^d, \end{cases}$$

where $|\nabla| = (-\Delta)^{\frac{1}{2}}$, $d \geq 3$, $1 < \alpha \leq 2$, and $F(u)$ is the nonlinear term of Hartree type given by

$$F(u) = \mu(|\cdot|^{-2\alpha} * |u|^2)u, \quad \mu \in \mathbb{R} \setminus \{0\}.$$

Fractional Schrödinger equation appears in fractional quantum mechanics (see [32, 33, 34]), where Laskin generalized the Brownian-like quantum mechanical path, in the Feynman path integral approach to quantum mechanics, to the α -stable Lévy-like quantum mechanical path.

The solution u of (1.1) formally satisfies the mass and energy conservation laws:

$$(1.2) \quad \begin{aligned} m(u) &= \|u(t)\|_{L^2}^2, \\ E(u) &= K(u) + P(u), \end{aligned}$$

where

$$K(u) = \frac{1}{2} \langle u, |\nabla|^\alpha u \rangle, \quad P(u) = \frac{1}{4} \langle u, \mu(|x|^{-2\alpha} * |u|^2)u \rangle.$$

Here $\langle \cdot, \cdot \rangle$ is the complex inner product in L^2 . Hence $H^{\frac{\alpha}{2}}$ is referred to energy space.

The equation (1.1) has a scaling invariance. In fact, if u is a solution of (1.1), then for any $\lambda > 0$ the scaled function u_λ given by

$$u_\lambda(t, x) = \lambda^{-\frac{\alpha}{2} + \frac{d}{2}} u(\lambda^\alpha t, \lambda x)$$

is also a solution. Since $\dot{H}^{\frac{\alpha}{2}}$ -norm is preserved under the scaling $u \mapsto u_\lambda$, (1.1) is said to be energy-critical if $s = \frac{\alpha}{2}$. It is also said to be super(sub)-critical if $s < \frac{\alpha}{2}$ ($s > \frac{\alpha}{2}$, respectively).

By Duhamel's formula, (1.1) is written as an integral equation

$$(1.3) \quad u = U(t)\phi - i\mu \int_0^t U(t-t')(|\cdot|^{-2\alpha} * |u(t')|^2)u(t') dt'.$$

Here we define the linear propagator $U(t)f$ to be the solution to the linear problem $i\partial_t z = |\nabla|^\alpha z$ with initial datum f . Then it is formally given by

$$(1.4) \quad U(t)f = \mathcal{F}^{-1}e^{-it|\xi|^\alpha}\mathcal{F}f = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi - t|\xi|^\alpha} \widehat{f}(\xi) d\xi,$$

where $\widehat{f} = \mathcal{F}f$ denotes the Fourier transform of f such that $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx$ and we denote its inverse Fourier transform by $\mathcal{F}^{-1}g(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi) d\xi$.

For the linear propagator $U(t)$, Strichartz estimate is known to hold

Lemma 1.1 (Theorem 2 in [21]). *Let $d \geq 2$ and $2/q + d/r = d/2$, $2 \leq q, r \leq \infty$.*

$$\| |\nabla|^{-\frac{2-\alpha}{q}} U(t)f \|_{L_T^q L^r} \lesssim \|f\|_{L_x^2}.$$

The implicit constant does not depend on $T > 0$.

Here and after $L_T^q X$ denotes mixed normed space $L^q([-T, T]; X(\mathbb{R}^d))$ for a Banach space X on \mathbb{R}^d and $L_t^q X L^q(\mathbb{R}; X)$. Due to the weak dispersion of $U(t)$ the estimate accompanies a derivative loss of order $2-\alpha$. But if one imposes radial assumptions or angularly regular condition on f , then a derivative loss can be recovered and even a regularity gain can be obtained (see [18, 26]).

Using Lemma 1.1, the local well-posedness of (1.1) can be shown in the subcritical case ($s > \frac{\alpha}{2}$). Actually a little revision of Proposition 4.1 in [16] gives

Proposition 1.2. *Let $s > \frac{\alpha}{2}$. If $\phi \in H^s$ then there exists a positive time T such that (1.1) has a unique solution $u \in C([-T, T]; H^s) \cap L_T^2(H_{2d/(d-2)}^{s-(2-\alpha)/2})$.*

On the other hand, when $s = \frac{\alpha}{2}$, by using radial Strichartz estimate, the local and small data global well-posedness to (1.1) are proven under the radial assumption of ϕ as follows.

Proposition 1.3 (Theorem 5.2 in [16]). *Let $\frac{2d}{2d-1} \leq \alpha \leq 2$ and $\phi \in H_{rad}^{\frac{\alpha}{2}}$. then there exists a positive time T such that (1.1) has a unique solution $u \in C([-T, T]; H^{\frac{\alpha}{2}}) \cap L_T^3 H_r^{\frac{\alpha}{2}}$, $r = \frac{2n}{n-2\alpha}$. If $\|\phi\|_{\dot{H}^{\frac{\alpha}{2}}}$ is sufficiently small, then (1.1) is globally well-posed.*

Recently the author and collaborators of [20] obtained global well-posedness for $\frac{2d}{2d-1} < \alpha < 2$ without smallness when $\mu > 0$ and with $\|\phi\|_{\dot{H}^{\frac{\alpha}{2}}} < \|W_\alpha\|_{\dot{H}^{\frac{\alpha}{2}}}$ when $\mu < 0$, where W_α is a steady state solution of (1.1). Also See [25] for power type. In [29] a power type case was treated in some critical regularity without radial assumption. When $\alpha = 2$, the equation is much easier to handle, so there exist numerous well-posed and ill-posedness results (see [13, 38, 27, 45, 41, 39, 36, 37]).

In this paper we focus on supercritical case ($s < \frac{\alpha}{2}$). Many dispersive equations are known to be ill-posed in supercritical regime (see [1, 8, 9, 12, 22]). For fractional Schrödinger equation, we also observe some negative results. One can readily show the following with a slight modification of illposedness in [15, 28]. So we omit the proof.

Proposition 1.4. *If $s < \alpha(1 - \frac{\alpha}{2})$ and the flow map $\phi \mapsto u$ exists in a small neighborhood of the origin as a map from $H^s(\mathbb{R}^d)$ to $C([-T, T]; H^s)$, then it fails to be C^3 at the origin.*

Nonetheless, using probabilistic arguments, Bourgain [6], Burq-Tzvetkov [10, 11], Colliander-Oh [23] and Bényi-Oh-Pocovnicu [3, 4] established positive results on subsets of H^s for the supercritical case (see also [44, 24, 40, 7, 42, 43, 35]). Especially, in [3, 4], the authors introduced a randomization for functions in the usual Sobolev space on \mathbb{R}^d .

Many of these works are on the dispersive equation with power type nonlinearity. So we concern the Cauchy problem with random initial data of the equation with Hartree nonlinearity. Because of nonlocal nonlinearity, the problem is more complicated. More precisely, we cannot apply Hölder inequality and bilinear Strichartz estimates (Lemma 3.6 and Lemma 3.7) directly. In order to overcome the difficulty, we decompose functions with respect to frequency as in [36]. Now we state our main theorem.

Theorem 1.5. *Let $\max(\frac{2\alpha-1}{4\alpha-3} \cdot \frac{\alpha}{2}, \frac{1}{2}) < s < \frac{\alpha}{2}$ and $\phi \in H^s$. Consider randomization ϕ^ω defined in (2.2) with a probability space (Ω, \mathcal{F}, P) satisfying the condition (2.1). Then (1.1) is almost surely locally wellposed in the sense that there exists C, c, γ and $\sigma = \frac{\alpha}{2} +$ such that for each $T \ll 1$, there exists a set $\Omega_T \subset \Omega$ with the following properties:*

- (1) $P(\Omega \setminus \Omega_T) \leq C \exp\left(-\frac{c}{T^\gamma \|\phi\|_{H^s}^2}\right)$
- (2) *For each $\omega \in \Omega_T$, there exists a unique solution $u \in C([0, T]; H^s)$ to (1.1) with initial data ϕ^ω .*
- (3) *Duhamel part of the solution is smoother than initial data, i.e*

$$u - U(t)\phi^\omega \in C([0, T]; H^\sigma).$$

The rest of the paper is organized as follows: In Section 2, we briefly review randomization adapted to the Wiener decomposition in [3, 4]. And in section 3, we introduce Bourgain space $X^{s,b}$ and show bilinear Strichartz estimates. Lastly in section 4, we shall prove Theorem 1.1.

2. RANDOMIZATION

We briefly review randomization adapted to the Wiener decomposition in [3, 4]. Let $\psi \in \mathcal{S}$ be a function satisfying

$$\text{supp } \psi \subset [-1, 1]^d \text{ and } \sum_{n \in \mathbb{Z}^d} \psi(\xi - n) = 1.$$

And we define pseudo-differential operator $\psi(D - n)$ as a Fourier multiplier

$$\psi(D - n)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \psi(\xi - n) \widehat{u}(\xi) d\xi.$$

Then given a function $f \in L^2(\mathbb{R}^d)$, we have

$$f = \sum_{n \in \mathbb{Z}^d} \psi(D - n)f.$$

Let $\{g_n\}_{n \in \mathbb{Z}^d}$ be a sequence of independent mean zero complex-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where the real and imaginary parts of g_n are independent and endowed with probability distribution μ_n^1 and μ_n^2 . We assume there exists $c > 0$ such that

$$(2.1) \quad \left| \int_{\mathbb{R}} e^{\gamma x} d\mu_n^j \right| \leq e^{c\gamma^2},$$

for all $\gamma \in \mathbb{R}, n \in \mathbb{Z}^d, j = 1, 2$. Thereafter we define Wiener randomization of f by

$$(2.2) \quad f^\omega := \sum_{n \in \mathbb{Z}^d} g_n(\omega) \psi(D - n) f.$$

We recall several well-known useful probabilistic estimates.

Lemma 2.1 (Lemma 3.1 in [10]). *For given $\{c_n\} \in \ell^2(\mathbb{Z}^d)$ and $p \geq 2$, there exists $C > 0$ such that*

$$\left\| \sum_{n \in \mathbb{Z}^d} g_n(\omega) c_n \right\|_{L^p(\Omega)} \leq C \sqrt{q} \|c_n\|_{\ell_n^2(\mathbb{Z}^d)}.$$

Lemma 2.2 (Lemma 2.2 in [3]). *Given $f \in H^s(\mathbb{R}^d)$, we have for any $\lambda > 0$,*

$$P\left(\|f^\omega\|_{H^s(\mathbb{R}^d)} > \lambda\right) \leq C e^{-c\lambda^2 \|f\|_{H^s}^{-2}}.$$

Lemma 2.3 (Lemma 2.3 in [3]). *Given $f \in L^2(\mathbb{R}^d)$ and finite $p \geq 2$, there exists $C, c > 0$ such that for any $\lambda > 0$,*

$$P\left(\|f^\omega\|_{L^p(\mathbb{R}^d)} > \lambda\right) \leq C e^{-c\lambda^2 \|f\|_{L^2}^{-2}}.$$

In particular, f^ω is in $L^p(\mathbb{R}^d)$ almost surely.

Exactly same arguments for Schrödinger equation in [3, 4] give probabilistic Strichartz estimates for fractional Schrödinger equation. Actually the only property of linear propagator used in those papers is that L^2 -norm of linear propagator is conserved in time (see Proposition 1.3 in [3]).

Proposition 2.4. *Given $f \in L^2(\mathbb{R}^d)$, let f^ω be its randomization. Then, given $2 \leq q, r \leq \infty$, for all $T > 0$ and $\lambda > 0$ there exists $C, c > 0$ such that*

$$(2.3) \quad P(\|U(t)f^\omega\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} > \lambda) \leq C \exp\left(-c \frac{\lambda^2}{T^{\frac{2}{q}} \|f\|_{L^2}^2}\right).$$

3. BOURGAIN SPACE

We introduce Bourgain space $X^{s,b}$ defined as follows: for $s, b \in \mathbb{R}$

$$X^{s,b} = \left\{ \varphi \in \mathcal{S}' : \|\varphi\|_{X^{s,b}} := \|\langle \xi \rangle^s \langle \tau - |\xi|^2 \rangle^b \tilde{\varphi}(\tau, \xi)\|_{L^2(\mathbb{R} \times \mathbb{R}^d)} < \infty \right\},$$

where $\langle a \rangle = 1 + |a|$ and $\tilde{\varphi}$ denotes the time-space Fourier transform. In what follows we mention a few of well-known properties of $X^{s,b}$ space.

Lemma 3.1. *Let $T \in (0, 1)$ and $b \in (\frac{1}{2}, \frac{3}{2}]$. Then for $s \in \mathbb{R}$ and $\theta \in [0, \frac{3}{2} - b]$ the following hold*

$$\begin{aligned} \|\eta_T(t)U(t)f\|_{X^{s,b}(\mathbb{R} \times \mathbb{R}^d)} &\lesssim T^{\frac{1}{2}-b}\|f\|_{H^s(\mathbb{R}^d)}, \\ \|\eta_T(t) \int_0^t U(t-t')\eta_T(t')F(t')dt'\|_{X^{s,b}(\mathbb{R} \times \mathbb{R}^d)} &\lesssim T^\theta\|F\|_{X^{s,b-1+\theta}(\mathbb{R} \times \mathbb{R}^d)}. \end{aligned}$$

Lemma 3.2. *Let (q, r) satisfy $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, and $(d, q, r) \neq (2, 2, \infty)$. Then for $b > \frac{1}{2}$ we have*

$$\|u\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|u\|_{X^{\frac{2-\alpha}{q}, b}(\mathbb{R} \times \mathbb{R}^d)}.$$

The above lemma follows from Strichartz estimates (Lemma 1.1). By interpolation with trivial estimate $\|u\|_{L_{t,x}^2} \lesssim \|u\|_{X^{0,0}}$, we have the following lemma.

Lemma 3.3. *Let $q \geq 2$. Then for $b > \frac{1}{2}$ we have*

$$\|u\|_{L_t^q L_x^2} \lesssim \|u\|_{X^{0, b(1-\frac{2}{q})}(\mathbb{R} \times \mathbb{R}^d)}.$$

Because of scaling symmetry, Strichartz estimate is optimal. But if one considers interaction of two different frequency localized data, it is possible to obtain bilinear Strichartz estimate.

Lemma 3.4 (Lemma 2.2 in [17]). *Let $d \geq 2$. Suppose that $\text{supp } \widehat{f} \subset A(N_1)$ and $\text{supp } \widehat{g} \subset A(N_2)$ with $N_1 \leq N_2$. Then we have*

$$\|U(t)fU(t)g\|_{L_{t,x}^2} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{d+\alpha-2}{4}} (N_1 N_2)^{\frac{d-\alpha}{4}} \|f\|_{L_x^2} \|g\|_{L_x^2} = N_1^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}} \|f\|_{L_x^2} \|g\|_{L_x^2}.$$

Moreover, we prove bilinear estimates for data whose Fourier support in a small ball.

Lemma 3.5. *Let $d \geq 2$. Suppose that $\text{supp } \widehat{f} \subset B(\xi_0, \rho_1)$, with $\rho_1, |\xi_0| \ll 1$ and $\text{supp } \widehat{g} \subset A(1)$. Then we have*

$$\|U(t)fU(t)g\|_{L_{t,x}^2} \lesssim \rho_1^{\frac{d-1}{2}} \|f\|_{L_x^2} \|g\|_{L_x^2}.$$

Proof. By decomposing the Fourier support of g into finite number of sets, rotation and mild dilation, it suffices to prove the estimates when $\text{supp } \widehat{g} \subset B(e_1, \delta)$ for some $0 < \delta \ll 1$. By definition of $U(t)$, we have

$$U(t)f(x)U(t)g(x) = (2\pi)^{-2d} \int e^{i(x \cdot (\xi + \eta) - t(|\xi|^\alpha + |\eta|^\alpha))} \widehat{f}(\xi) \widehat{g}(\eta) d\xi d\eta.$$

For each $\bar{\xi} = (\xi_2, \dots, \xi_d)$, we define a bilinear operator

$$B_{\bar{\xi}}(f, g) = \int_{\mathbb{R}^{1+d}} e^{i(x \cdot (\xi + \eta) - t(|\xi|^\alpha + |\eta|^\alpha))} \widehat{f}(\xi_1, \bar{\xi}) \widehat{g}(\eta) d\xi_1 d\eta.$$

We make the change of variable $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_{d+1}) = (\xi + \eta, |\xi|^\alpha + |\eta|^\alpha)$ with the observation $|\frac{\partial \zeta}{\partial(\xi_1, \eta)}| = \alpha|\xi_1||\xi|^{\alpha-2} - \eta_1|\eta|^{\alpha-2} \sim 1$. Then applying Plancherel's theorem and reversing the change variables ($\zeta \rightarrow (\xi_1, \eta)$), we get

$$\|B_{\bar{\xi}}(f, g)\|_{L_t^2 L_x^2} \lesssim \|\widehat{f}(\xi_1, \bar{\xi}) \widehat{g}(\eta)\|_{L_{\xi_1, \eta}^2}.$$

Hence by Mikowski's inequality, we have

$$\|U(t)fU(t)g\|_{L_t^2 L_x^2} = \left\| \int B_{\bar{\xi}}(f, g) d\bar{\xi} \right\|_{L_t^2 L_x^2} \lesssim \int \|\widehat{f}(\xi_1, \bar{\xi}) \widehat{g}(\eta)\|_{L_{\xi_1, \eta}^2} d\bar{\xi} \lesssim \rho_1^{\frac{d-1}{2}} \|f\|_{L_x^2} \|g\|_{L_x^2}.$$

The last inequality follows from the fact that Fourier support of f is in $B(\xi_0, \rho_1)$. \square

From Lemma 3.4, Lemma 3.5 and definition of $X^{s,b}$ space, one can prove the following lemma.

Lemma 3.6. *Let $d \geq 2$. Consider $u, v \in X^{0,b}$ for $b > \frac{1}{2}$. Then we have*

(1) *If $\text{supp } \widehat{u} \subset A(N_1)$ and $\text{supp } \widehat{v} \subset A(N_2)$ with $N_1 \leq N_2$, then*

$$\|uv\|_{L_{t,x}^2} \lesssim N_1^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}} \|u\|_{X^{0,b}} \|v\|_{X^{0,b}}.$$

(2) *If $\text{supp } \widehat{u} \subset B(\xi_0, N)$ and $\text{supp } \widehat{v} \subset A(N_2)$ with $N, |\xi_0| \ll N_2$, then*

$$\|uv\|_{L_{t,x}^2} \lesssim N^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}} \|u\|_{X^{0,b}} \|v\|_{X^{0,b}}.$$

Furthermore interpolation with trivial inequality $\|uv\|_{L_{t,x}^2} \lesssim \|u\|_{L_{t,x}^\infty} \|v\|_{L_{t,x}^2} \lesssim \|u\|_{X^{\frac{d}{2}+\frac{1}{2}+}} \|v\|_{X^{0,0}}$ yields the following useful lemma.

Lemma 3.7. *Let $d \geq 2$. Then, for given small $\varepsilon > 0$, we have*

(1) *If $\text{supp } \widehat{u} \subset A(N_1)$ and $\text{supp } \widehat{v} \subset A(N_2)$ with $N_1 \leq N_2$, then*

$$\|uv\|_{L_{t,x}^2} \lesssim N_1^{\frac{d-1}{2}+2\varepsilon} N_2^{\frac{1-\alpha}{2}+2\varepsilon} \|u\|_{X^{0,\frac{1}{2}+\varepsilon}} \|v\|_{X^{0,\frac{1}{2}-\varepsilon}}.$$

(2) *If $\text{supp } \widehat{u} \subset B(\xi_0, N)$ and $\text{supp } \widehat{v} \subset A(N_2)$ with $|\xi_0| \sim N_1$ and $N, N_1 \ll N_2$, then*

$$\|uv\|_{L_{t,x}^2} \lesssim N^{\frac{d-1}{2}-\varepsilon_1} N_1^{\varepsilon_2} N_2^{\frac{1-\alpha}{2}+2\varepsilon} \|u\|_{X^{0,\frac{1}{2}+\varepsilon}} \|v\|_{X^{0,\frac{1}{2}-\varepsilon}}$$

where $\varepsilon_1 = \frac{2(d-1)\varepsilon}{1+2\varepsilon}$ and $\varepsilon_2 = \frac{2(d+2\varepsilon)\varepsilon}{1+2\varepsilon}$ so that $\varepsilon_2 - \varepsilon_1 = 2\varepsilon$.

4. ALMOST SURE LOCAL WELLPOSEDNESS

We will prove Theorem 1.5. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its randomization. We concern (1.1) with ϕ^ω . Let $z(t) := U(t)\phi^\omega$ and $v(t) := u(t) - U(t)\phi^\omega$. Then (1.1) becomes

$$(4.1) \quad \begin{cases} i\partial_t v = |\nabla|^\alpha v + F(v+z), & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ v(0, x) = 0. \end{cases}$$

By Duhamel's principle, (4.1) is written as integral equation

$$v(t) = \int_0^t U(t-t') F(v+z)(t') dt'.$$

Let η be a smooth cutoff function supported on $[-2, 2]$, $\eta = 1$ on $[-1, 1]$, and let $\eta_T(t) = \eta(t/T)$.

Then we have

$$v(t) = \eta_T(t) \int_0^t U(t-t') \eta_T(t') F(\eta_T(t') v + \eta_T(t') z)(t') dt'.$$

So we define \mathcal{D} by

$$\mathcal{D}v(t) = \eta_T(t) \int_0^t U(t-t') \eta_T(t') F(\eta_T(t') v + \eta_T(t') z)(t') dt'.$$

Now it suffices to prove \mathcal{D} has a fixed point in closed subset of $C_t H_x^s([0, T] \times \mathbb{R}^d)$ outside a set of probability $\leq C \exp\left(-\frac{c}{T^\gamma \|\phi\|_{H^s}^2}\right)$. For that purpose, we show contraction inequality (Proposition 4.1) for \mathcal{D} . Then exactly same arguments in p.11 of [3] give Theorem 1.1.

Proposition 4.1. *Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its randomization. Then, there exists $\sigma = 1+$, $b = \frac{1}{2}+$ and $\theta = 0+$ such that for each small $T \ll 1$ and $R > 0$, we have*

$$\begin{aligned} \|Dv\|_{X^{\sigma,b}} &\leq C_1 T^\theta (\|v\|_{X^{\sigma,b}}^3 + R^3) \\ \|Dv - Dw\|_{X^{\sigma,b}} &\leq C_2 T^\theta (\|v\|_{X^{\sigma,b}}^2 + \|w\|_{X^{\sigma,b}}^2 + R^2) \|v - w\|_{X^{\sigma,b}}, \end{aligned}$$

outside a set of probability at most $C \exp\left(-c \frac{R^2}{\|\phi\|_{H^s}^2}\right)$.

Proof. We shall only show first estimate, then second estimate can be also proven similarly. By using Lemma 3.1 and duality, we get

$$\begin{aligned} \|Dv(t)\|_{X^{\sigma,b}} &\lesssim T^\theta \|F(\eta_T v + \eta_T z)\|_{X^{\sigma,b-1+\theta}} \\ &= T^\theta \sup_{\|v_4\|_{X^{0,1-b-\theta}} \leq 1} \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma [F(\eta_T v + \eta_T z)] v_4 dx dt \right|. \end{aligned}$$

So there exist 6 terms to be considered

$$\begin{aligned} (1) & \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * |\eta_T v|^2) \eta_T v v_4 dx dt \right| \\ (2) & \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * |\eta_T z|^2) \eta_T z v_4 dx dt \right| \\ (3) & \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (\eta_T v \overline{\eta_T z} + \overline{\eta_T v} \eta_T z)) \eta_T v v_4 dx dt \right| \\ (4) & \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * |\eta_T v|^2) \eta_T z v_4 dx dt \right| \\ (5) & \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (\eta_T v \overline{\eta_T z} + \overline{\eta_T v} \eta_T z)) \eta_T z v_4 dx dt \right| \\ (6) & \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * |\eta_T z|^2) \eta_T v v_4 dx dt \right|. \end{aligned}$$

We will estimate each term by using Strichartz estimates, Bilinear Strichartz estimates and probabilistic estimates.

1st Term : vvv term

$$(4.2) \quad \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * |\eta_T v|^2) \eta_T v v_4 dx dt \right|.$$

By Hölder inequality, (4.2) is bounded by

$$\|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (|\eta_T v|^2) \eta_T v)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^\varepsilon L_x^2}.$$

for some small positive ε such that $\varepsilon < \frac{1}{\alpha}(\sigma - \frac{\alpha}{2})$. From Lemma 3.3, we have

$$\|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \lesssim \|v_4\|_{X^{0, \tilde{b}(1-2\varepsilon)}} = \|v_4\|_{X^{0, 1-b-\theta}}.$$

In order to deal with nonlocal term, we introduce useful lemmas.

Lemma 4.2 (Lemma A1 \sim Lemma A4 in [30]). *For any $s \geq 0$ we have*

$$\| |\nabla|^s (uv) \|_{L^r} \lesssim \| |\nabla|^s u \|_{L^{q_1}} \|v\|_{L^{q_2}} + \|u\|_{L^{q_1}} \| |\nabla|^s v \|_{L^{r_2}},$$

where $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{r_2}$, $r_i \in (1, \infty)$, $q_i \in (1, \infty]$, $i = 1, 2$.

Lemma 4.3 (Lemma 3.2 in [19]). *For any $0 < \varepsilon_1 < d - 2\alpha$ we have*

$$\| |x|^{-2\alpha} * (|u|^2) \|_{L^\infty} \lesssim \|u\|_{L^{\frac{2d}{d-2\alpha-\varepsilon_1}}} \|u\|_{L^{\frac{2d}{d-2\alpha+\varepsilon_1}}}.$$

By using Lemma 4.2, we get

$$\begin{aligned} \| \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (|\eta_T v|^2) \eta_T v) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} &\lesssim \| |x|^{-2\alpha} * (|\eta_T v|^2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \| \langle \nabla \rangle^\sigma \eta_T v \|_{L_t^\infty L_x^2} \\ &+ \| \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * |\eta_T v|^2) \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \| \eta_T v \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}. \end{aligned}$$

Thereafter, from Lemma 3.2, we obtain

$$\| \langle \nabla \rangle^\sigma \eta_T v \|_{L_t^\infty L_x^2} \lesssim \|v\|_{X^{\sigma, b}}.$$

For $\| |x|^{-2\alpha} * (|\eta_T v|^2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty}$, we use Lemma 4.3 and Hölder inequality

$$\| |x|^{-2\alpha} * (|\eta_T v|^2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \lesssim \| \eta_T v \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \| \eta_T v \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}.$$

Then, from Sobolev embedding, we obtain

$$\| \eta_T v \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \lesssim \| \langle \nabla \rangle^{\sigma_1} v \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2(1-\varepsilon)}}}, \quad \| \eta_T v \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \lesssim \| \langle \nabla \rangle^{\sigma_2} v \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2(1-\varepsilon)}}},$$

where $\sigma_1 = \frac{d-2(1-\varepsilon)}{2} - \frac{d-2\alpha-\alpha\varepsilon}{2}$ and $\sigma_2 = \frac{d-2(1-\varepsilon)}{2} - \frac{d-2\alpha+\alpha\varepsilon}{2}$. And Lemma 3.2 yield

$$\| \langle \nabla \rangle^{\sigma_1} v \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2(1-\varepsilon)}}}, \| \langle \nabla \rangle^{\sigma_2} v \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2(1-\varepsilon)}}} \lesssim \|v\|_{X^{\sigma, b}},$$

because $\varepsilon < \frac{1}{\alpha}(\sigma - \frac{\alpha}{2})$ gives $\sigma_2 + (2 - \alpha) \cdot \frac{1-\varepsilon}{2} < \sigma_1 + (2 - \alpha) \cdot \frac{1-\varepsilon}{2} < \sigma$.

For $\| \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * |\eta_T v|^2) \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}}$, we use fractional integration Theorem

$$\| \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * |\eta_T v|^2) \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \lesssim \| \langle \nabla \rangle^\sigma (|\eta_T v|^2) \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}}.$$

Then Lemma 4.2 and Hölder inequality give

$$\| \langle \nabla \rangle^\sigma (|\eta_T v|^2) \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \lesssim \| \langle \nabla \rangle^\sigma v \|_{L_t^\infty L_x^2} \|v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}.$$

By using Solobev inequality and Lemma 3.2, we have

$$\|v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \lesssim \| \langle \nabla \rangle^{\sigma_2} v \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2(1-\varepsilon)}}} \lesssim \|v\|_{X^{\sigma, b}}.$$

In conclusion, we get (4.2) is bounded by $\|v\|_{X^{\sigma,b}}^3 \|v_4\|_{X^{0,1-b-\theta}}$.

In order to handle remaining terms, we make dyadic decomposition and assume Fourier transform of z_i , v_i is supported on the set $\{\xi \sim N_i\}$. In dealing with 2nd, 4th and 6th terms, we may assume $N_1 \leq N_2$.

2nd Term : zzz term

$$(4.3) \quad \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * z_1 z_2) z_3 v_4 dx dt \right|.$$

We consider two cases separately

- i. $\max(N_1, N_2, N_3) \sim \text{med}(N_1, N_2, N_3)$
- ii. $\max(N_1, N_2, N_3) \gg \text{med}(N_1, N_2, N_3)$.

Case (2.i) : $\max(N_1, N_2, N_3) \sim \text{med}(N_1, N_2, N_3)$

By Hölder inequality, (4.3) is bounded by

$$\|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2)) z_3\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2}$$

for some small positive ε . From Lemma 3.3, we have

$$\|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \lesssim \|v_4\|_{X^{0,\tilde{b}(1-2\varepsilon)}} = \|v_4\|_{X^{0,1-b-\theta}}.$$

And by using Lemma 4.2, we get

$$\begin{aligned} \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2)) z_3\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} &\lesssim \| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \\ &+ \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}} \|z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}}. \end{aligned}$$

We first concern term $\| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}}$. Fractional integration theorem and Hölder inequality yield

$$\| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}} \lesssim \|z_1 z_2\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{d-4\alpha}}} \lesssim \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}}.$$

Then from $\max(N_1, N_2, N_3) \sim \text{med}(N_1, N_2, N_3)$, we obtain

$$\begin{aligned} &\| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \\ &\lesssim \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|\langle \nabla \rangle^{\frac{\sigma}{2}} z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|\langle \nabla \rangle^{\frac{\sigma}{2}} z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \\ &+ \|\langle \nabla \rangle^{\frac{\sigma}{2}} z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|\langle \nabla \rangle^{\frac{\sigma}{2}} z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}}. \end{aligned}$$

For $\|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}}$, we use fractional integration Theorem

$$\|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}} \lesssim \|\langle \nabla \rangle^\sigma (z_1 z_2)\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{3d-4\alpha}}}.$$

Then fractional Leibniz rule and Hölder inequality give

$$\|\langle \nabla \rangle^\sigma (z_1 z_2)\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{3d-4\alpha}}} \lesssim \|\langle \nabla \rangle^\sigma z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} + \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}}.$$

And from $\max(N_1, N_2, N_3) \sim \text{med}(N_1, N_2, N_3)$, we have

$$\begin{aligned} & \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}} \|z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \\ & \lesssim \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|\langle \nabla \rangle^{\frac{\sigma}{2}} z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|\langle \nabla \rangle^{\frac{\sigma}{2}} z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \\ & + \|\langle \nabla \rangle^{\frac{\sigma}{2}} z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|\langle \nabla \rangle^{\frac{\sigma}{2}} z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}}. \end{aligned}$$

Therefore, from Proposition 2.4 and $\frac{\sigma}{2} < s$, we conclude that

$$\sum_{\max(N_1, N_2, N_3) \sim \text{med}(N_1, N_2, N_3)} (4.3) \lesssim R^3 \|v_4\|_{X^{0,1-b-\theta}}$$

outside a set of probability

$$\leq C \exp \left(-c \frac{R^2}{T^{\frac{2-2\varepsilon}{3}} \|\phi\|_{H^s}^2} \right).$$

Case (2.ii) : $\max(N_1, N_2, N_3) \gg \text{med}(N_1, N_2, N_3)$

Since the case of $\max(N_1, N_2, N_3) \sim N_2$ can be similarly handled, we only deal with the case of $\max(N_1, N_2, N_3) \sim N_3$. Then we consider 4 cases separately.

- a. $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1$
- b. $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1$
- c. $N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$
- d. $N_4 \sim N_3 \gg N_2, N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}}.$

Subcase (2.ii.a) : $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1$

The spatial Fourier support of $z_1 z_2$ is contained in $A(2N_2)$. So we note that $|\nabla|^{2\alpha-d} \sim N_2^{2\alpha-d}$ on the spatial Fourier support of $z_1 z_2$. Then we have

$$\frac{|\nabla|^{2\alpha-d}}{N_2^{\alpha-d}} (z_1 z_2) = (2\pi)^{-d} \int \int e^{ix \cdot (\xi + \eta)} \chi\left(\frac{\xi}{N_2}\right) \left(\frac{|\xi + \eta|}{N_2}\right)^{2\alpha-d} \chi\left(\frac{\eta}{N_2}\right) \widehat{z}_1(\xi) \widehat{z}_2(\eta) d\xi d\eta,$$

where χ is supported in $B(0, 1)$. Now we take Fourier series expansion for $\Psi(\xi, \eta) = \chi(\xi)|\xi + \eta|^{2\alpha-d} \chi(\eta)$ on the cube of side length 2π which contains the support of Ψ to get

$$\chi(\xi)|\xi + \eta|^{2\alpha-d} \chi(\eta) = \sum_{k, l \in \mathbb{Z}^d} C_{k, l} e^{i(k \cdot \xi + l \cdot \eta)}$$

with $\sum_{k,l} |C_{k,l}| \leq C$. Then we have the identity

$$\frac{|\nabla|^{\alpha-d}}{N_1^{\alpha-d}}(z_1 z_2) = \sum_{k,l \in \mathbb{Z}^d} C_{k,l} z_1^k z_2^l,$$

where $z_1^k = (2\pi)^{-d} \int e^{ix \cdot \xi} e^{ik \cdot \xi} \widehat{z_1}(\xi) d\xi$ and $z_2^l = (2\pi)^{-d} \int e^{ix \cdot \eta} e^{il \cdot \eta} \widehat{z_2}(\eta) d\eta$.

So we need to estimate

$$\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (z_1^k z_2^l z_3) v_4| dx dt.$$

And since $|\xi + \eta|^\alpha \lesssim |\xi|^\alpha + |\eta|^\alpha$, it suffices to deal with

$$\begin{aligned} & N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma z_1^k z_2^l z_3 v_4| dx dt, \quad N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l z_3 v_4| dx dt \\ & \text{and } N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt. \end{aligned}$$

Third term will be only considered, because remaining two terms can be handled similarly.

By using Hölder inequality, Lemma 3.6 and Lemma 3.7, we get

$$\begin{aligned} & N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \lesssim N_2^{2\alpha-d} \|z_1^k \langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^2} \|z_2^l v_4\|_{L_{t,x}^2} \\ & \lesssim N_1^{\frac{d-1}{2}} N_2^{2\alpha-d+\frac{d-1}{2}+2\varepsilon} N_3^{1-\alpha+\sigma+2\varepsilon} \|z_1^k\|_{X^{0,b}} \|z_2^l\|_{X^{0,b}} \|z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $\|z_1^k\|_{X^{0,b}} = \|z_1\|_{X^{0,b}}$, $\|z_2^l\|_{X^{0,b}} = \|z_2\|_{X^{0,b}}$ and $\sum_{k,l} |C_{k,l}| \leq C$, we have

$$\begin{aligned} & \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma ((z_1^k z_2^l) z_3) v_4| dx dt \\ & \lesssim N_1^{\frac{d-1}{2}} N_2^{2\alpha-d+\frac{d-1}{2}+2\varepsilon} N_3^{1-\alpha+\sigma+2\varepsilon} \|z_1\|_{X^{0,b}} \|z_2\|_{X^{0,b}} \|z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

And by using Bernstein inequality carrying out summation in N_1 , we get

$$\begin{aligned} & \sum_{N_1 \ll N_2} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma ((z_1^k z_2^l) z_3) v_4| dx dt \\ & \lesssim N_2^{2\alpha-1-2s+2\varepsilon} N_3^{\alpha-1+\sigma-s+2\varepsilon} \|z\|_{X^{s,b}} \|z\|_{X^{s,b}} \|z_3\|_{X^{s,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1$ and $2\alpha-1+s+2\varepsilon > 0$, from summation in N_2 , we obtain

$$\sum_{N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1} (4.3) \lesssim N_3^{-2s\frac{\alpha-1}{2\alpha-1}+\sigma-s+\frac{6\alpha-4}{2\alpha-1}\varepsilon} \|z\|_{X^{s,b}} \|z\|_{X^{s,b}} \|z_3\|_{X^{s,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}.$$

In order to make summation in N_3 be finite, the power $-2s\frac{\alpha-1}{2\alpha-1}+\sigma-s$ should be negative. So we need the condition

$$s > \sigma \frac{2\alpha-1}{4\alpha-3}.$$

After carrying out summation in N_3 and applying Lemma 3.1, we have

$$\sum_{N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1} (4.3) \lesssim T^{0-} \|\phi^\omega\|_{H^s}^3 \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}.$$

Therefore, from Lemma 2.2, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1} (4.3) \lesssim T^{0-} R^3 \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}$$

outside a set of probability

$$C \exp\left(-c \frac{R^2}{\|\phi\|_{H^s}^2}\right).$$

Subcase (2.ii.b) : $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1$.

This case is more delicate because $|\nabla|^{2\alpha-d}$ might be singular on Fourier support of $z_1 z_2$. First we decompose $|\nabla|^{2\alpha-d}$ such that

$$|\nabla|^{2\alpha-d} = \sum_N N^{2\alpha-d} \psi(|\nabla|/N),$$

with a cut-off ψ supported in $A(1)$. Here $\psi(|\nabla|)$ is pseudo-differential operator defined by $\psi(|\nabla|)f = \mathcal{F}^{-1}(\psi(|\cdot|)\mathcal{F}f)$. Then we have

$$\int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * z_1 z_2) z_3 v_4 dx dt = \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma \left(\sum_{-\infty}^{N_2} N^{2\alpha-d} \psi(|\nabla|/N) (z_1 z_2) z_3 v_4 \right) dx dt.$$

After that we decompose z_1 and z_2 into functions having Fourier supports in cubes of side length $2^{-2}N$. Let $\{Q\}$ be a collection of essentially disjoint cubes of side length $2^{-2}N$ covering $A(N_2)$. Let us define z_{iQ} by $\widehat{z_{iQ}} = \chi_Q(\xi) \widehat{z_i}$ for $i = 1, 2$. Then we have $z_i = \sum_Q z_{iQ}$ for $i = 1, 2$. Since $N_1 \sim N_2$, we may restrict $Q \subset A(N_2)$. So, we have

$$\begin{aligned} & \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (N^{2\alpha-d} \psi(|\nabla|/N) (z_1 z_2) z_3 v_4) dx dt \\ & \lesssim \sum_{Q, Q'} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (N^{2\alpha-d} \psi(|\nabla|/N) (z_{1Q} z_{2Q'}) z_3 v_4) dx dt \\ & = \sum_{\text{dist}(Q, -Q') \leq 4N} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (N^{2\alpha-d} \psi(|\nabla|/N) (z_{1Q} z_{2Q'}) z_3 v_4) dx dt. \end{aligned}$$

Here, the last equality follows from the fact that $\psi(|\nabla|/N) (z_{1Q} z_{2Q'}) = 0$ if $\text{dist}(Q, -Q') > 4N$.

We observe that

$$\psi(|\nabla|/N) (z_{1Q} z_{2Q'}) = \int \int e^{ix \cdot (\xi + \eta)} \chi(\xi/N - \xi_0) \psi((\xi + \eta)/N) \chi(\eta/N - \eta_0) \widehat{z_{1Q}} \widehat{z_{2Q'}} d\xi d\eta$$

for some $\xi_0, \eta_0 \in \mathbb{R}^d$ and χ supported in $B(0, 1)$. Let us take the Fourier series expansion for $\Psi(\xi, \eta) = \chi(\xi - \xi_0) \psi(\xi + \eta) \chi(\eta - \eta_0)$ on the cube of side length 2π which contains the support

of Ψ to get

$$\chi(\xi - \xi_0)\psi(\xi + \eta)\chi(\eta - \eta_0) = \sum_{k,l \in \mathbb{Z}^d} C_{k,l} e^{i(k \cdot \xi + l \cdot \eta)}$$

with $\sum_{k,l} |C_{k,l}| \leq C$, independent of ξ_0, η_0 . So, we have

$$\psi(|\nabla|/N)(z_{1Q} z_{2Q'}) = \sum_{k,l \in \mathbb{Z}^d} C_{k,l} z_{1Q}^k z_{2Q'}^l$$

where $z_{1Q}^k = \int e^{2\pi i x \cdot \xi} e^{2\pi i k \cdot \xi} \widehat{z_{1Q}}(\xi) d\xi$ and $z_{2Q'}^l = \int e^{2\pi i x \cdot \eta} e^{2\pi i l \cdot \eta} \widehat{z_{2Q'}}(\eta) d\eta$. Hence we obtain

$$\begin{aligned} & \sum_{\text{dist}(Q, -Q') \leq 4N} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (N^{2\alpha-d} \psi(|\nabla|/N)(z_{1Q} z_{2Q'}) z_3) v_4| dx dt \\ &= \sum_{\text{dist}(Q, -Q') \leq 4N} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (N^{2\alpha-d} (\sum_{k,l \in \mathbb{Z}^d} C_{k,l} z_{1Q}^k z_{2Q'}^l) z_3) v_4| dx dt \\ &\lesssim \sum_{\text{dist}(Q, -Q') \leq 4N} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma ((z_{1Q}^k z_{2Q'}^l) z_3) v_4| dx dt. \end{aligned}$$

So we need to handle

$$\begin{aligned} & N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma z_{1Q}^k z_{2Q'}^l z_3 v_4| dx dt, \quad N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_{1Q}^k \langle \nabla \rangle^\sigma z_{2Q'}^l z_3 v_4| dx dt \\ & \text{and } N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_{1Q}^k z_{2Q'}^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt. \end{aligned}$$

Third term will be only considered, because remaining two terms can be handled similarly. By using Hölder inequality, Lemma 3.6 and Lemma 3.7, we get

$$\begin{aligned} & N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_{1Q}^k z_{2Q'}^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \lesssim N^{2\alpha-d} \|z_{1Q}^k \langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^2} \|z_{2Q'}^l v_4\|_{L_{t,x}^2} \\ & \lesssim N^{2\alpha-1-\varepsilon_1} N_2^{\varepsilon_2} N_3^{1-\alpha+\sigma+2\varepsilon} \|z_{1Q}^k\|_{X^{0,b}} \|z_{2Q'}^l\|_{X^{0,b}} \|z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $\|z_{1Q}^k\|_{X^{0,b}} = \|z_{1Q}\|_{X^{0,b}}$, $\|z_{2Q'}^l\|_{X^{0,b}} = \|z_{2Q'}\|_{X^{0,b}}$ and $\sum_{k,l} |C_{k,l}| \leq C$, we have

$$\begin{aligned} & \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_{1Q}^k z_{2Q'}^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ & \lesssim N^{2\alpha-1-\varepsilon_1} N_2^{\varepsilon_2} N_3^{1-\alpha+\sigma+2\varepsilon} \|z_{1Q}\|_{X^{0,b}} \|z_{2Q'}\|_{X^{0,b}} \|z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Thereafter we use Cauchy-Schwartz inequality, orthogonality and Bernstein inequality to get

$$\begin{aligned} & \sum_{\text{dist}(Q, -Q') \leq 4N} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_{1Q}^k z_{2Q'}^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ & \lesssim N^{2\alpha-1-\varepsilon_1} N_2^{\varepsilon_2} N_3^{1-\alpha+\sigma+2\varepsilon} \|z_1\|_{X^{0,b}} \|z_2\|_{X^{0,b}} \|z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \\ & \lesssim N^{2\alpha-1-\varepsilon_1} N_2^{-2s+\varepsilon_2} N_3^{1-\alpha+\sigma-s+2\varepsilon} \|z_1\|_{X^{s,b}} \|z_2\|_{X^{s,b}} \|z_3\|_{X^{s,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Then summation in $N \lesssim N_2$ gives

$$\begin{aligned} & \sum_{N=-\infty}^{N_2} \sum_{\text{dist}(Q, -Q') \leq 4N} \sum_{k, l \in \mathbb{Z}^d} C_{k, l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ & \lesssim N_2^{2\alpha-1-2s+2\varepsilon} N_3^{1-\alpha+\sigma-s+2\varepsilon} \|z_1\|_{X^{s,b}} \|z_2\|_{X^{s,b}} \|z_3\|_{X^{s,b}} \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1$ and $2\alpha-1-2s+2\varepsilon > 0$, we have

$$\begin{aligned} & \sum_{N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1} \sum_{N=-\infty}^{N_2} \sum_{\text{dist}(Q, -Q') \leq 4N} \sum_{k, l \in \mathbb{Z}^d} C_{k, l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma ((z_1^k z_2^l z_3) v_4)| dx dt \\ & \lesssim N_3^{-2s \frac{\alpha-1}{2\alpha-1} + \sigma - s + \frac{6\alpha-4}{2\alpha-1} \varepsilon} \|z\|_{X^{s,b}} \|z\|_{X^{s,b}} \|z_3\|_{X^{s,b}} \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}. \end{aligned}$$

Thus, from Lemma 3.1, we obtain

$$\lesssim T^{0-} N_3^{-2s \frac{\alpha-1}{2\alpha-1} + \sigma - s + \frac{6\alpha-4}{2\alpha-1} \varepsilon} \|\phi^\omega\|_{H^s}^2 \|P_{N_3} \phi^\omega\|_{H^s} \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}.$$

We can carry out summation in N_3 , because the power $(-2s \frac{\alpha-1}{2\alpha-1} + \sigma - s)$ is negative.

Therefore, by using Lemma 2.2, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1} (4.3) \lesssim T^{0-} R^3 \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}$$

outside a set of probability

$$C \exp \left(-c \frac{R^2}{\|\phi\|_{H^s}^2} \right).$$

Subcase (2.ii.c) $N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$

Adopting method in Case (2.ii.a), it suffices to estimate

$$\sum_{k, l \in \mathbb{Z}^d} C_{k, l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (z_1^k z_2^l z_3) v_4| dx dt.$$

So we have to handle

$$\begin{aligned} & N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma z_1^k z_2^l z_3 v_4| dx dt, \quad N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l z_3 v_4| dx dt \\ & \text{and } N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt. \end{aligned}$$

Third term will be only considered, because remaining two terms can be handled similarly. By using Hölder inequality, Lemma 3.6 and Lemma 3.7, we get

$$\begin{aligned} & N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \lesssim N_2^{2\alpha-d} \|z_2^l\|_{L_{t,x}^4} \|\langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^4} \|z_1^k v_4\|_{L_{t,x}^2} \\ & \lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|z_1^k\|_{X^{0,b}} \|z_2^l\|_{L_{t,x}^4} \|\langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}} \end{aligned}$$

Thereafter, from $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|z_1^k\|_{X^{0,b}} = \|z_1\|_{X^{0,b}}, \|z_2^l\|_{L_{t,x}^4} = \|z_2\|_{L_{t,x}^4}$), Bernstein inequality and Lemma 3.1, we obtain

$$\begin{aligned} & \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ & \lesssim T^{0-} N_2^{-s+2\alpha-d} N_3^{\sigma-s} N_1^{\frac{d-1}{2}+2\varepsilon-s} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|P_{N_1} \phi^\omega\|_{H^s} \|\langle \nabla \rangle^s z_2\|_{L_{t,x}^4} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Then summation in N_1 and N_2 yields

$$\begin{aligned} & \sum_{N_2=N_3^{\frac{\alpha-1}{2\alpha-1}}}^{N_3} \sum_{N_1=1}^{N_3^{\frac{\alpha-1}{2\alpha-1}}} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ & \lesssim T^{0-} N_3^{-s \left(\frac{4\alpha-3}{2\alpha-1} \right) + \frac{\alpha-1}{2(2\alpha-1)}(2\alpha-d)+\sigma+\frac{6\alpha-4}{2\alpha-1}\varepsilon} \|\phi^\omega\|_{H^s} \|\langle \nabla \rangle^s z_2\|_{L_{t,x}^4} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Hence, from Lemma 2.2 and Lemma 2.4, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1} (4.3) \lesssim T^{0-} R^3 \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}$$

outside a set of probability

$$C \exp \left(-c \frac{R^2}{\|\phi\|_{H^s}^2} \right) + C \exp \left(-c \frac{R^2}{T^{\frac{1}{2}} \|\phi\|_{H^s}^2} \right).$$

Subcase (2.ii.d) : $N_4 \sim N_3 \gg N_2 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}}$

Hölder inequality and Lemma 3.3 yield (4.3) is bounded by

$$\lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2)) z_3\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2)) z_3\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{X^{0,1-b-\theta}}$$

for some small positive ε . Thereafter, from Fractional integration theorem, Hölder inequality and Lemma 4.2, we obtain

$$\begin{aligned} (4.3) & \lesssim \|\langle \nabla \rangle^\sigma z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|v_4\|_{X^{0,1-b-\theta}} \\ & + \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|v_4\|_{X^{0,1-b-\theta}} \\ & + \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|v_4\|_{X^{0,1-b-\theta}}. \end{aligned}$$

Then we use Bernstein's inequality and carry out summation in N_1, N_2 to get

$$\sum_{N_2=N_1}^{N_3} \sum_{N_1=N_3^{\frac{\alpha-1}{2\alpha-1}}}^{N_2} (4.3) \lesssim N_3^{\sigma-\frac{4\alpha-3}{2\alpha-1}s} \|\langle \nabla \rangle^s z\|^2_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|\langle \nabla \rangle^s z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{2d}{d-\frac{4\alpha}{3}}}} \|v_4\|_{X^{0,1-b-\theta}}.$$

Therefore, from Lemma 2.4, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_2 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}}} (4.3) \lesssim R^3 \|v_4\|_{X^{0,1-b-\theta}}$$

outside a set of probability

$$\leq C \exp \left(-c \frac{R^2}{T^{\frac{2-2\varepsilon}{3}} \|\phi\|_{H^s}^2} \right).$$

3rd Term : vzv term

$$(4.4) \quad \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (v_1 z_2) v_3) v_4 dx dt \right|.$$

We consider two cases separately

- i. $\max(N_1, N_3) \gtrsim N_2$
- ii. $\max(N_1, N_3) \ll N_2 \sim N_4$.

Case (3.i) : $\max(N_1, N_3) \gtrsim N_2$

We assume $N_1 \geq N_3$, because the other case can be similarly handled. Hölder inequality, Lemma 4.2 and Lemma 3.3 yield

$$\begin{aligned} (4.4) &\lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (v_1 z_2) v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\ &\lesssim \| |x|^{-2\alpha} * (v_1 z_2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|\langle \nabla \rangle^\sigma v_3\|_{L_t^\infty L_x^2} \|v_4\|_{X^{0,1-b-\theta}} \\ &\quad + \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (v_1 z_2))\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|v_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}}. \end{aligned}$$

Thereafter we use Lemma 4.3, Hölder inequality, Sobolev embedding and Lemma 4.2 to get

$$\begin{aligned} &\lesssim \left(\|v_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|v_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \right. \\ &\quad \left. + \|\langle \nabla \rangle^\sigma (v_1 z_2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \right) \times \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}}. \end{aligned}$$

Then Lemma 4.2 and Lemma 3.2 give

$$\begin{aligned} &\lesssim \|v_1\|_{X^{\sigma,b}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}} \\ &\quad + \|\langle \nabla \rangle^\sigma v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}} \\ &\quad + \|v_1\|_{L_t^\infty L_x^2} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}}. \end{aligned}$$

Since $N_1 \gtrsim N_2$, we have

$$\begin{aligned} &\|v_1\|_{L_t^\infty L_x^2} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}} \\ &\lesssim \|\langle \nabla \rangle^\sigma v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}}. \end{aligned}$$

Therefore, from Lemma 2.4, we conclude that

$$\sum_{\max(N_1, N_3) \gtrsim N_2} (4.4) \lesssim R \|v\|_{X^{\sigma, b}}^2 \|v_4\|_{X^{0, 1-b-\theta}}$$

outside a set of probability

$$\leq C \exp \left(-c \frac{R^2}{T^{1-\varepsilon} \|\phi\|_{H^{0+}}} \right).$$

Case (3. ii) : $\max(N_1, N_3) \ll N_2 \sim N_4$.

We assume $N_3 \geq N_1$, because other case can be handled similarly. As in Case (2.ii.a), we shall deal with

$$\sum_{k, l \in \mathbb{Z}^d} C_{k, l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (v_1^k z_2^l v_3) v_4| dx dt.$$

So we need to estimate

$$\begin{aligned} & N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma v_1^k z_2^l v_3 v_4| dx dt, \quad N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt \\ & \text{and } N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma v_3 v_4| dx dt. \end{aligned}$$

We will consider second term only, because remaining two terms can be handled similarly. By using Hölder inequality, Lemma 3.6 and Lemma 3.7, we get

$$\begin{aligned} & N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt \lesssim N_2^{2\alpha-d} \|\langle \nabla \rangle^\sigma z_2^l\|_{L_{t,x}^4} \|v_3\|_{L_{t,x}^4} \|v_1^k v_4\|_{L_{t,x}^2} \\ & \lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1^k\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_2^l\|_{L_{t,x}^4} \|v_3\|_{L_{t,x}^4} \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}} \end{aligned}$$

Thereafter we use $\sum_{k, l} |C_{k, l}| \leq C$ (with $\|v_1^k\|_{X^{0,b}} = \|v_1\|_{X^{0,b}}, \|z_2^l\|_{L_{t,x}^4} = \|z_2\|_{L_{t,x}^4}$), Bernstein inequality and Lemma 3.2 to obtain

$$\begin{aligned} & N_2^{2\alpha-d} \sum_{k, l \in \mathbb{Z}^d} C_{k, l} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt \\ & \lesssim N_2^{\sigma-s+2\alpha-d} N_1^{\frac{d-1}{2}+2\varepsilon-\sigma} N_3^{-\sigma+\frac{d-\alpha}{4}} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_2\|_{L_{t,x}^4} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}. \end{aligned}$$

Then we carry out summation in N_1

$$\begin{aligned} & \sum_{N_1=1}^{N_3} N_2^{2\alpha-d} \sum_{k, l \in \mathbb{Z}^d} C_{k, l} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt \\ & \lesssim N_2^{\sigma-s+2\alpha-d+\frac{1-\alpha}{2}+2\varepsilon} N_3^{\frac{d-1}{2}+2\varepsilon-\sigma-\sigma+\frac{d-\alpha}{4}} \|v\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_2\|_{L_{t,x}^4} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}. \end{aligned}$$

We observe that $-d+2\alpha+\frac{1}{2} < s$ gives $\sigma-s+2\alpha-d+\frac{1-\alpha}{2}+2\varepsilon < 0$, when $\sigma-\frac{\alpha}{2}, \varepsilon$ is sufficiently small. So if $\frac{d-1}{2}-2\sigma+\frac{d-\alpha}{4}+2\varepsilon < 0$, then summation can be carried out over N_2, N_3 . Otherwise

it should be checked that $\sigma - s + 2\alpha - d + \frac{1-\alpha}{2} + 2\varepsilon + \frac{d-1}{2} - 2\sigma + \frac{d-\alpha}{4} + 2\varepsilon < 0$. Actually the following hold

$$\begin{aligned} & \sigma - s + 2\alpha - d + \frac{1-\alpha}{2} + 2\varepsilon + \frac{d-1}{2} - 2\sigma + \frac{d-\alpha}{4} + 2\varepsilon \\ &= -\sigma - s + \frac{5}{4}\alpha - \frac{d}{4} + 4\varepsilon < -\left(\frac{6\alpha-4}{4\alpha-3}\right)\sigma + \frac{5}{4}\alpha - \frac{d}{4} + 4\varepsilon \quad (\text{because } s > \frac{2\alpha-1}{4\alpha-3}\sigma) \\ &< -\left(\frac{6\alpha-4}{4\alpha-3}\right)\sigma + \frac{3}{4}\alpha + 4\varepsilon \quad (\text{because } d > 2\alpha) < -\frac{1}{4(4\alpha-3)}\alpha + 4\varepsilon \quad (\text{because } \sigma > \frac{\alpha}{2}) < 0 \end{aligned}$$

when ε is sufficiently small. Hence we have that

$$\sum_{\max(N_1, N_3) \ll N_2 \sim N_4} (4.4) \lesssim \|v\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z\|_{L_{t,x}^4} \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}.$$

Therefore, from Lemma 2.3, we conclude that

$$\sum_{\max(N_1, N_3) \ll N_2 \sim N_4} (4.4) \lesssim R \|v\|_{X^{\sigma,b}}^2 \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}$$

outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{1}{2}} \|\phi\|_{H^s}^2}\right).$$

4th Term : vvz term

$$(4.5) \quad \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * v_1 v_2) z_3 v_4 dx dt \right|.$$

We consider 3 cases separately

- i. $N_2 \gtrsim N_3$
- ii. $N_1 \ll N_2 \ll N_3 \sim N_4$
- iii. $N_1 \sim N_2 \ll N_3 \sim N_4$.

Case (4.i) : $N_2 \gtrsim N_3$

Hölder inequality and Lemma 3.3 yield

$$\begin{aligned} (4.5) &\lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (v_1 v_2) z_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\ &\lesssim \| |x|^{-2\alpha} * (v_1 v_2) \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \\ &\quad + \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (v_1 v_2))\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \end{aligned}$$

Then we use Sobolev embedding and Hölder inequality to get

$$\begin{aligned} &\lesssim \|v_1 v_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \\ &\quad + \|\langle \nabla \rangle^\sigma (v_1 v_2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}}. \end{aligned}$$

And Lemma 4.2 gives

$$\begin{aligned} &\lesssim \|v_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|v_2\|_{L_t^\infty L_x^2} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \\ &\quad + \|v_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^\sigma v_2\|_{L_t^\infty L_x^2} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \\ &\quad + \|\langle \nabla \rangle^\sigma v_1\|_{L_t^\infty L_x^2} \|v_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \end{aligned}$$

Therefore, from $N_2 \gtrsim N_3$, Lemma 3.2 and Lemma 2.4, we conclude that

$$\sum_{N_2 \gtrsim N_3} (4.5) \lesssim R \|v\|_{X^{\sigma,b}}^2 \|v_4\|_{X^{0,1-b-\theta}}$$

outside a set of probability

$$\leq C \exp \left(-c \frac{R^2}{T^{1-\varepsilon} \|\phi\|_{H^{0+}}} \right).$$

Case (4. ii) : $N_1 \ll N_2 \ll N_3 \sim N_4$

Adopting method in Case (2.ii.a), we need to deal with

$$\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (v_1^k v_2^l z_3) v_4| dx dt.$$

So we have to estimate

$$\begin{aligned} &N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma v_1^k v_2^l z_3 v_4| dx dt, \quad N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma v_2^l z_3 v_4| dx dt \\ &\quad \text{and } N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k v_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt. \end{aligned}$$

Third term will be only considered, because remaining two terms can be handled similarly. By using Hölder inequality and Lemma 3.7, we get

$$\begin{aligned} &N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k v_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \lesssim N_2^{2\alpha-d} \|v_2^l\|_{L_{t,x}^4} \|\langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^4} \|v_1^k v_4\|_{L_{t,x}^2} \\ &\lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1^k\|_{X^{0,b}} \|v_2^l\|_{L_{t,x}^4} \|\langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Then $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|v_1^k\|_{X^{0,b}} = \|v_1\|_{X^{0,b}}$, $\|v_2^l\|_{L_{t,x}^4} = \|v_2\|_{L_{t,x}^4}$), Bernstein inequality and Lemma 3.2 yield

$$\begin{aligned} &\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k v_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ &\lesssim N_2^{-\sigma+\frac{d-\alpha}{4}+2\alpha-d} N_1^{\frac{d-1}{2}+2\varepsilon-\sigma} N_3^{\sigma-s} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1\|_{X^{\sigma,b}} \|v_2\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Thereafter we carry out summation in N_1 to get

$$\begin{aligned} &\sum_{N_1=1}^{N_2} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k v_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ &\lesssim N_2^{-2\sigma-\frac{d}{4}+\frac{7}{4}\alpha-\frac{1}{2}+2\varepsilon} N_3^{\sigma-s+\frac{1-\alpha}{2}+2\varepsilon} \|v\|_{X^{\sigma,b}} \|v_2\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Here we observe that $-2\sigma - \frac{d}{4} + \frac{7}{4}\alpha - \frac{1}{2} + 2\varepsilon < \frac{\alpha}{4} - \frac{1}{2} + 2\varepsilon < 0$ when ε is sufficiently small. Hence one can carry out summation over N_2 . For summation over N_3 , it is necessary that $\sigma - s + \frac{1-\alpha}{2} + 2\varepsilon < 0$, which is true when $s > \frac{1}{2}$ and ε is sufficiently small.

Therefore, from Lemma 2.4, we conclude that

$$\sum_{N_1 \ll N_2 \ll N_3 \sim N_4} (4.5) \lesssim \|v\|_{X^{\sigma,b}}^2 \|\langle \nabla \rangle^s z\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \lesssim R \|v\|_{X^{\sigma,b}}^2 \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}$$

outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{1}{2}} \|\phi\|_{H^s}^2}\right).$$

Case (4.iii) : $N_1 \sim N_2 \ll N_3 \sim N_4$

As in Case (2.ii.b), we consider

$$\sum_{\text{dist}(Q, -Q') \leq 4N} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma ((v_{1Q}^k v_{2Q'}^l) z_3) v_4| dx dt.$$

So we have to estimate

$$N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma v_{1Q}^k v_{2Q'}^l z_3 v_4| dx dt, \quad N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_{1Q}^k \langle \nabla \rangle^\sigma v_{2Q'}^l z_3 v_4| dx dt$$

and $N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_{1Q}^k v_{2Q'}^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt.$

Third term will be only considered, because remaining two terms can be handled similarly. By using Hölder inequality, Lemma 3.6 and Lemma 3.7, we get

$$\begin{aligned} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_{1Q}^k v_{2Q'}^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt &\lesssim N^{2\alpha-d} \|v_{2Q'}^l \langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^2} \|v_{1Q}^k v_4\|_{L_{t,x}^2} \\ &\lesssim N^{2\alpha-d} N^{d-1+2\varepsilon} N_3^{1-\alpha+2\varepsilon} \|v_{1Q}^k\|_{X^{0,b}} \|v_{2Q'}^l\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|v_{1Q}^k\|_{X^{0,b}} = \|v_{1Q}\|_{X^{0,b}}$, $\|v_{2Q'}^l\|_{X^{0,b}} = \|v_{2Q'}\|_{X^{0,b}}$), we have

$$\begin{aligned} &\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_{1Q}^k v_{2Q'}^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ &\lesssim N^{2\alpha-d} N^{d-1+2\varepsilon} N_3^{1-\alpha+2\varepsilon} \|v_{1Q}\|_{X^{0,b}} \|v_{2Q'}\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Thereafter we use Cauchy-Schwartz inequality, orthogonality and Bernstein inequality to obtain

$$\begin{aligned} &\sum_{\text{dist}(Q, -Q') \leq 4N} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma ((v_{1Q}^k v_{2Q'}^l) z_3) v_4| dx dt \\ &\lesssim N^{2\alpha-d} N^{d-1+2\varepsilon} N_3^{1-\alpha+2\varepsilon} \|v_1\|_{X^{0,b}} \|v_2\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \\ &\lesssim N^{2\alpha-1+2\varepsilon} N_2^{-2\sigma} N_3^{1-\alpha+\sigma-s+2\varepsilon} \|v_1\|_{X^{\sigma,b}} \|v_2\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $N \leq N_1 \sim N_2 \ll N_4$, we have

$$\begin{aligned} & \sum_{N_2=1}^{N_3} \sum_{N=-\infty}^{N_2} \sum_{\text{dist}(Q, -Q') \leq 4N} \sum_{k, l \in \mathbb{Z}^d} C_{k, l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma ((v_{1Q}^k v_{2Q'}^l) z_3) v_4| dx dt \\ & \lesssim N_3^{\alpha-1-\sigma-s+4\varepsilon} \|v\|_{X^{\sigma, b}} \|v\|_{X^{\sigma, b}} \|\langle \nabla \rangle^s z_3\|_{X^{0, b}} \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}} \end{aligned}$$

Then, from Lemma 3.1, we get

$$\lesssim T^{0-} N_3^{\alpha-1-\sigma-s+4\varepsilon} \|P_{N_3} \phi^\omega\|_{H^s} \|v\|_{X^{\sigma, b}} \|v\|_{X^{\sigma, b}} \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}.$$

Because $\alpha - 1 - \sigma - s + 2\varepsilon_1 < 0$, we have finite summation of N_3 .

Hence, by using Lemma 2.1, we conclude that

$$\sum_{N_1 \sim N_2 \ll N_3 \sim N_4} (4.5) \lesssim T^{0-} R \|v\|_{X^{\sigma, b}}^2 \|v_4\|_{X^{0, \frac{1}{2}-\varepsilon}}$$

outside a set of probability

$$C \exp \left(-c \frac{R^2}{\|\phi\|_{H^s}^2} \right).$$

5th Term : vzz term

$$(4.6) \quad \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (v_1 z_2)) z_3 v_4 dx dt \right|.$$

We consider 4 cases separately

- i. $N_1 \gtrsim \max(N_2, N_3)$
- ii. $N_1 \ll N_2 \sim N_3$
- iii. $N_1, N_2 \ll N_3$
- iv. $N_1, N_3 \ll N_2$.

Case (5.i) : $N_1 \gtrsim \max(N_2, N_3)$.

Hölder inequality, Lemma 4.2 and Lemma 3.3 give

$$\begin{aligned} (4.6) & \lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (v_1 z_2) z_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\ & \lesssim \| |x|^{-2\alpha} * (v_1 z_2) \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0, 1-b-\theta}} \\ & \quad + \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (v_1 z_2))\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0, 1-b-\theta}}. \end{aligned}$$

Thereafter we use Sobolev embedding to obtain

$$\begin{aligned} & \lesssim \|v_1 z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0, 1-b-\theta}} \\ & \quad + \|\langle \nabla \rangle^\sigma (v_1 z_2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0, 1-b-\theta}} \end{aligned}$$

Then Hölder inequality and Lemma 4.2 yield

$$\begin{aligned} &\lesssim \|v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \\ &\quad + \|v_1\|_{L_t^\infty L_x^2} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \\ &\quad + \|\langle \nabla \rangle^\sigma v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}}. \end{aligned}$$

Therefore, from Lemma 2.4, we conclude that

$$\sum_{N_1 \gtrsim \max(N_2, N_3)} (4.6) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}}$$

outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{1-\varepsilon} \|\phi\|_{H^{0+}}}\right).$$

Case (5.ii) : $N_1 \ll N_2 \sim N_3$

We assume $N_4 \geq N_1$, because other case can be handled similarly. As in Case (2.ii.a), we shall deal with

$$\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (v_1^k z_2^l z_3) v_4| dx dt.$$

So we need to handle

$$\begin{aligned} &N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma v_1^k z_2^l z_3 v_4| dx dt, \quad N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma z_2^l z_3 v_4| dx dt \\ &\text{and } N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt. \end{aligned}$$

Second term will be only considered, because remaining two terms can be handled similarly. By using Hölder inequality and Lemma 3.7, we get

$$\begin{aligned} &N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma z_2^l z_3 v_4| dx dt \lesssim N_2^{2\alpha-d} \|\langle \nabla \rangle^\sigma z_2^l\|_{L_{t,x}^4} \|z_3\|_{L_{t,x}^4} \|v_1^k v_4\|_{L_{t,x}^2} \\ &\lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1^k\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_2^l\|_{L_{t,x}^4} \|z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \end{aligned}$$

Then from $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|v_1^k\|_{X^{0,b}} = \|v_1\|_{X^{0,b}}$, $\|z_2^l\|_{L_{t,x}^4} = \|z_2\|_{L_{t,x}^4}$) and Bernstein inequality, we obtain

$$\begin{aligned} &\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma z_2^l z_3 v_4| dx dt \\ &\lesssim N_2^{\sigma-s+2\alpha-d} N_1^{\frac{d-1}{2}+2\varepsilon-\sigma} N_3^{-s} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_2\|_{L_{t,x}^4} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \\ &\lesssim N_2^{\sigma-s+2\alpha-d} N_1^{\frac{d-1}{2}-\sigma+\frac{1-\alpha}{2}+4\varepsilon} N_3^{-s} \|v_1\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_2\|_{L_{t,x}^4} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Last line follows from $N_1 \leq N_4$.

Thereafter we carry out summation in N_1

$$\sum_{N_1=1}^{N_2} (4.6) \lesssim \begin{cases} N_2^{\frac{3\alpha-d}{2}-2s+4\varepsilon} \|v\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_2\|_{L_{t,x}^4} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \\ \quad (\text{ when } \frac{d-1}{2} - \sigma + \frac{1-\alpha}{2} + 4\varepsilon > 0) \\ N_2^{\sigma-2s+2\alpha-d} \|v\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_2\|_{L_{t,x}^4} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \\ \quad (\text{ when } \frac{d-1}{2} - \sigma + \frac{1-\alpha}{2} + 4\varepsilon \leq 0). \end{cases}$$

Since $\frac{3\alpha-d}{2} - 2s + 4\varepsilon \leq \frac{\alpha}{2} - 2s + 4\varepsilon < 0$ for sufficiently small ε and $\sigma - 2s + 2\alpha - d < 0$, we have

$$\sum_{N_1 \ll N_2 \sim N_3} (4.6) \lesssim \|v\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z\|_{L_{t,x}^4}^2 \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}.$$

Therefore, from Lemma 2.4, we conclude that

$$\sum_{N_1 \ll N_2 \sim N_3} (4.6) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}$$

outside a set of probability

$$\leq C \exp \left(-c \frac{R^2}{T^{\frac{1}{2}} \|\phi\|_{H^s}^2} \right).$$

Case (5.iii) : $N_1, N_2 \ll N_3$

We consider 5 cases separately

- a. $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1$
- b. $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1$
- c. $N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$
- d. $N_4 \sim N_3 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2$
- e. $N_4 \sim N_3 \gg N_2, N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}}.$

Subcase (5.iii.a) : $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1$

Similarly to Case (2.ii.a), we need to estimate

$$\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (v_1^k z_2^l z_3) v_4| dx dt.$$

Hence we shall deal with

$$\begin{aligned} & N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma v_1^k z_2^l z_3 v_4| dx dt, \quad N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma z_2^l z_3 v_4| dx dt \\ & \text{and } N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt. \end{aligned}$$

Third term will be only considered, because remaining two terms can be handled similarly. By using Hölder inequality, Lemma 3.6 and Lemma 3.7, we have

$$\begin{aligned} & N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \lesssim N_2^{2\alpha-d} \|v_1^k \langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^2} \|z_2^l v_4\|_{L_{t,x}^2} \\ & \lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}} N_3^{\frac{1-\alpha}{2}} N_2^{\frac{d-1}{2}+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1^k\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_3\|_{X^{0,b}} \|z_2^l\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Then $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|v_1^k\|_{X^{0,b}} = \|v_1\|_{X^{0,b}}$, $\|z_2^l\|_{X^{0,b}} = \|z_2\|_{X^{0,b}}$), Bernstein inequality and Lemma 3.1 yield

$$\begin{aligned} & \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ & \lesssim N_1^{\frac{d-1}{2}-\sigma} N_3^{\frac{1-\alpha}{2}+\sigma-s} N_2^{\frac{d-1}{2}+2\alpha-d-s+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1\|_{X^{\sigma,b}} \|z_2\|_{X^{s,b}} \|z_3\|_{X^{s,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \\ & \lesssim T^{0-} N_1^{\frac{d-1}{2}-\sigma} N_3^{\frac{1-\alpha}{2}+\sigma-s} N_2^{\frac{d-1}{2}+2\alpha-d-s+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1\|_{X^{\sigma,b}} \prod_{j=2}^3 \|P_{N_j} \phi^\omega\|_{H^s} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Thereafter we carry out summation in N_1, N_2

$$\begin{aligned} & \sum_{N_2=1}^{N_3^{\frac{\alpha-1}{2\alpha-1}}} \sum_{N_1=1}^{N_2} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ & \lesssim T^{0-} N_3^{\sigma(\frac{\alpha}{2\alpha-1})-s(\frac{3\alpha-2}{2\alpha-1})+2\varepsilon(\frac{3\alpha-2}{2\alpha-1})} \|v\|_{X^{\sigma,b}} \|\phi^\omega\|_{H^s} \|P_{N_3} \phi^\omega\|_{H^s} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $s > \sigma \frac{2\alpha-1}{4\alpha-3} > \sigma \frac{\alpha}{3\alpha-2}$, summation in N_3 can be also carried out.

Therefore, from Lemma 2.2, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1} (4.6) \lesssim T^{0-} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}$$

outside a set of probability

$$C \exp(-c \frac{R^2}{\|\phi\|_{H^s}^2}).$$

Subcase (5.iii.b) : $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1$

Similarly to Case (2.ii.b), we consider

$$\sum_{\text{dist}(Q, -Q') \leq 4N} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (v_{1Q}^k z_{2Q'}^l z_3) v_4| dx dt.$$

So we need to handle

$$\begin{aligned} & N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma v_{1Q}^k z_{2Q'}^l z_3 v_4| dx dt, \quad N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_{1Q}^k \langle \nabla \rangle^\sigma z_{2Q'}^l z_3 v_4| dx dt \\ & \text{and } N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_{1Q}^k z_{2Q'}^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt. \end{aligned}$$

Third term will be only considered, because remaining two terms can be handled similarly.

Hölder inequality, Lemma 3.6 and Lemma 3.7 give

$$\begin{aligned} & N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_{1Q}^k z_{2Q'}^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \lesssim N^{2\alpha-d} \|v_{1Q}^k \langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^2} \|z_{2Q'}^l v_4\|_{L_{t,x}^2} \\ & \lesssim N^{2\alpha-d} N^{d-1+2\varepsilon} N_3^{1-\alpha+2\varepsilon} \|v_{1Q}^k\|_{X^{0,b}} \|z_{2Q'}^l\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

And from $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|v_{1Q}^k\|_{X^{0,b}} = \|v_{1Q}\|_{X^{0,b}}$, $\|z_{2Q'}^l\|_{X^{0,b}} = \|z_{2Q'}\|_{X^{0,b}}$), we have

$$\begin{aligned} & \sum_{k,l \in \mathbb{Z}^d} C_{k,l} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_{1Q}^k z_{2Q'}^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ & \lesssim N^{2\alpha-d} N^{d-1+2\varepsilon} N_3^{1-\alpha+2\varepsilon} \|v_{1Q}\|_{X^{0,b}} \|z_{2Q'}\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Then Cauchy-Schwartz inequality, orthogonality and Bernstein inequality yield

$$\begin{aligned} & \sum_{N=-\infty}^{N_2} \sum_{\text{dist}(Q,-Q') \leq 4N} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma ((v_{1Q}^k z_{2Q'}^l) z_3) v_4| dx dt \\ & \lesssim \sum_{N=-\infty}^{N_2} N^{2\alpha-d} N^{d-1+2\varepsilon} N_3^{1-\alpha+2\varepsilon} \|v_1\|_{X^{0,b}} \|z_2\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \\ & \lesssim N_2^{2\alpha-1+2\varepsilon} N_3^{1-\alpha+2\varepsilon} \|v_1\|_{X^{0,b}} \|z_2\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \\ & \lesssim N_2^{2\alpha-1-\sigma-s+2\varepsilon} N_3^{1-\alpha+\sigma-s+2\varepsilon} \|v_1\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_2\|_{X^{0,b}} \|\langle \nabla \rangle^s z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \end{aligned}$$

Since $N_2 \ll N_3^{\frac{\alpha-1}{2\alpha-1}}$ and $2\alpha-1-\sigma-s+\varepsilon_1 > 0$, we can carry out summation in N_2 so that

$$\begin{aligned} & \sum_{N_2=1}^{N_3^{\frac{\alpha-1}{2\alpha-1}}} \sum_{N=-\infty}^{N_2} \sum_{\text{dist}(Q,-Q') \leq 4N} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma ((v_{1Q}^k z_{2Q'}^l) z_3) v_4| dx dt \\ & \lesssim N_3^{\sigma(\frac{\alpha}{2\alpha-1})-s(\frac{3\alpha-2}{2\alpha-1})+2\varepsilon(\frac{3\alpha-2}{2\alpha-1})} \|v\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z\|_{X^{0,b}} \|\langle \nabla \rangle^s z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Then Lemma 3.1 yield

$$\lesssim T^{0-} N_3^{\sigma(\frac{\alpha}{2\alpha-1})-s(\frac{3\alpha-2}{2\alpha-1})+2\varepsilon(\frac{3\alpha-2}{2\alpha-1})} \|v\|_{X^{\sigma,b}} \|\phi^\omega\|_{H^s} \|P_{N_3} \phi^\omega\|_{H^s} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}.$$

Since $s > \sigma \frac{2\alpha-1}{4\alpha-3} > \sigma \frac{\alpha}{3\alpha-2}$, summation in N_3 can be carried out.

Therefore, from Lemma 2.2, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1} (4.6) \lesssim T^{0-} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}$$

outside a set of probability

$$C \exp(-c \frac{R^2}{\|\phi\|_{H^s}^2}).$$

Subcase (5.iii.c) : $N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$

As in Case (2.ii.a), we need to estimate

$$\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (v_1^k z_2^l z_3) v_4| dx dt.$$

So we have to deal

$$N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma v_1^k z_2^l z_3 v_4| dx dt, \quad N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma z_2^l z_3 v_4| dx dt$$

and $N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt.$

Third term will be only considered, because remaining two terms can be handled similarly. By using Hölder inequality and Lemma 3.7, we have

$$\begin{aligned} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt &\lesssim N_2^{2\alpha-d} \|z_2^l\|_{L_{t,x}^4} \|\langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^4} \|v_1^k v_4\|_{L_{t,x}^2} \\ &\lesssim N_1^{\frac{d-1}{2}-\sigma+2\varepsilon} N_3^{\sigma-s} N_2^{2\alpha-d-s} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1^k\|_{X^{\sigma,b}} \|z_2^l\|_{L_{t,x}^4} \|\langle \nabla \rangle^\sigma z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Then, from $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|v_1^k\|_{X^{\sigma,b}} = \|v_1\|_{X^{\sigma,b}}, \|z_2^l\|_{L_{t,x}^4} = \|z_2\|_{L_{t,x}^4}$) and Bernstein inequality, we obtain

$$\begin{aligned} &\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ &\lesssim T^{0-} N_1^{\frac{d-1}{2}-\sigma+2\varepsilon} N_3^{\sigma-s} N_2^{2\alpha-d-s} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1\|_{X^{\sigma,b}} \prod_{j=2}^3 \|\langle \nabla \rangle^s z_j\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$, $\frac{d-1}{2} - \sigma + \varepsilon_1 > 0$ and $2\alpha - d - s < 0$, we can carry out summation in N_1 and N_2 to get

$$\begin{aligned} &\sum_{N_2=N_3^{\frac{\alpha-1}{2\alpha-1}}}^{N_3} \sum_{N_1=1}^{N_3^{\frac{\alpha-1}{2\alpha-1}}} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ &\lesssim T^{0-} N_3^{\frac{(2\alpha-d)(\alpha-1)}{2(2\alpha-1)} + \sigma(\frac{\alpha}{2\alpha-1}) - s(\frac{3\alpha-2}{2\alpha-1}) + 2\varepsilon(\frac{3\alpha-2}{2\alpha-1})} \|v\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z\|_{L_{t,x}^4} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^4} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $\frac{(2\alpha-d)(\alpha-1)}{2(2\alpha-1)} + \sigma(\frac{\alpha}{2\alpha-1}) - s(\frac{3\alpha-2}{2\alpha-1}) + 2\varepsilon(\frac{3\alpha-2}{2\alpha-1}) < 0$ for sufficiently small ε , summation in N_3 can be also carried out.

Therefore, by using Lemma 2.4, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1} (4.6) \lesssim T^{0-} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}$$

outside a set of probability

$$C \exp\left(-c \frac{R^2}{T^{\frac{1}{2}} \|\phi\|_{H^s}^2}\right).$$

Subcase(5.iii.d) : $N_4 \sim N_3 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2$

Similarly to Case (2.ii.a), we need to handle

$$\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_1^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (v_1^k z_2^l z_3) v_4| dx dt.$$

So we shall estimate

$$N_1^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma v_1^k z_2^l z_3 v_4| dx dt, \quad N_1^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k \langle \nabla \rangle^\sigma z_2^l z_3 v_4| dx dt$$

and $N_1^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt.$

Third term will be only considered, because remaining two terms can be handled similarly. By using Hölder inequality and Lemma 3.7, we have

$$\begin{aligned} N_1^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt &\lesssim N_1^{2\alpha-d} \|v_1^k\|_{L_t^q L_x^r} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|z_2^l v_4\|_{L_{t,x}^2} \\ &\lesssim N_1^{2\alpha-d} N_2^{\frac{d-1}{2}-s+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1^k\|_{L_t^q L_x^r} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|z_2^l\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}, \end{aligned}$$

where $q = \frac{2(2-\alpha)}{3-2\alpha}$ when $d = 3$ and $q = 3$ when $d \geq 4$.

Then, from $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|v_1^k\|_{L_t^q L_x^r} = \|v_1\|_{L_t^q L_x^r}$, $\|z_2^l\|_{X^{0,b}} = \|z_2\|_{X^{0,b}}$) and Lemma 3.2, we obtain

$$\begin{aligned} &\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_1^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ &\lesssim N_1^{2\alpha-d+\frac{2-\alpha}{q}} N_2^{\frac{d-1}{2}-s+2\varepsilon} N_3^\sigma N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1\|_{X^{0,b}} \|z_2\|_{X^{0,b}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Thereafter we use Bernstein inequality and Lemma 3.1 to get

$$\begin{aligned} &\lesssim N_1^{2\alpha-d+\frac{2-\alpha}{q}-\sigma} N_2^{\frac{d-1}{2}-s+2\varepsilon} N_3^{\sigma-s} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1\|_{X^{\sigma,b}} \|z_2\|_{X^{s,b}} \|\langle \nabla \rangle^s z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \\ &\lesssim T^{0-} N_1^{2\alpha-d+\frac{2-\alpha}{q}-\sigma} N_2^{\frac{d-1}{2}-s+2\varepsilon} N_3^{\sigma-s} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_1\|_{X^{\sigma,b}} \|P_{N_2} \phi^\omega\|_{H^s} \|\langle \nabla \rangle^s z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $N_4 \sim N_3 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2$, we can carry out summation in N_1 and N_2 as follows

$$\begin{aligned} &\sum_{N_1=N_3^{\frac{\alpha-1}{2\alpha-1}}}^{N_3} \sum_{N_2=1}^{N_3^{\frac{\alpha-1}{2\alpha-1}}} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |v_1^k z_2^l \langle \nabla \rangle^\sigma z_3 v_4| dx dt \\ &\lesssim T^{0-} N_3^{\frac{\alpha-1}{2\alpha-1}(\frac{2-\alpha}{q}+\alpha-\frac{d}{2})+\frac{\alpha}{2\alpha-1}\sigma-\frac{3\alpha-2}{2\alpha-1}s+2\frac{3\alpha-2}{2\alpha-1}\varepsilon} \|v\|_{X^{\sigma,b}} \|\phi^\omega\|_{H^s} \|\langle \nabla \rangle^s z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \end{aligned}$$

After carrying out summation in N_3 , we apply Lemma 2.2 and Lemma 2.4 to get

$$\sum_{N_4 \sim N_3 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2} (4.6) \lesssim T^{0-} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}$$

outside a set of probability

$$C \exp(-c \frac{R^2}{\|\phi\|_{H^s}^2}) + C \exp(-c \frac{R^2}{T^{\frac{q-2}{q}} \|\phi\|_{H^s}^2}).$$

Subcase(5.iii.e) : $N_4 \sim N_3 \gg N_2, N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}}$

From Hölder inequality, Lemma 4.2 and Lemma 3.3, we obtain

$$\begin{aligned}
 (4.6) &\lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (v_1 z_2) z_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\
 &\lesssim \| |x|^{-2\alpha} * (v_1 z_2) \|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \\
 &\quad + \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (v_1 z_2))\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}}.
 \end{aligned}$$

And by using Sobolev embedding, we have

$$\begin{aligned}
 &\lesssim \|v_1 z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \\
 &\quad + \|\langle \nabla \rangle^\sigma (v_1 z_2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}}.
 \end{aligned}$$

Then Hölder inequality and Lemma 4.2 yield

$$\begin{aligned}
 &\lesssim \|v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \\
 &\quad + \|v_1\|_{L_t^\infty L_x^2} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}} \\
 &\quad + \|\langle \nabla \rangle^\sigma v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}}
 \end{aligned}$$

Thereafter we use Bernstein inequality and Lemma 3.2 to get

$$\lesssim N_1^{-\sigma} N_2^{-s} N_3^{\sigma-s} \|v_1\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^s z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}}$$

Now we carry out summation in N_1, N_2 and N_3

$$\sum_{N_4 \sim N_3 \gg N_2, N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}}} (4.6) \lesssim \|v\|_{X^{\sigma,b}} \|\langle \nabla \rangle^s z\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle \nabla \rangle^s z\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,1-b-\theta}}.$$

Therefore, from Lemma 2.4, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_2, N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}}} (4.6) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}}$$

outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{1-\varepsilon} \|\phi\|_{H^s}}\right).$$

6th Term : zzv term

$$(4.7) \quad \left| \int \int_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^\sigma (|x|^{-2\alpha} * z_1 z_2) v_3 v_4 dx dt \right|.$$

We consider 3 cases separately

- i. $N_3 \gtrsim N_2$
- ii. $N_3 \ll N_2 \sim N_1$
- iii. $N_3, N_1 \ll N_2$.

Case (6.i) : $N_3 \gtrsim N_2$.

Hölder inequality and Lemma 3.3 yield that (4.7) is bounded by

$$\|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2) v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2) v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{X^{0,1-b-\theta}},$$

for some small ε such that $0 < \varepsilon < \frac{1}{\alpha}(\sigma - \frac{\alpha}{2})$. Then we use Lemma 4.2 and Lemma 3.2 to get

$$\begin{aligned} & \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (|\eta_T v|^2) \eta_T v)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \\ & \lesssim \| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|\langle \nabla \rangle^\sigma v_3\|_{L_t^\infty L_x^2} + \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{L_t^\infty L_x^2} \\ & \lesssim \| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{X^{\sigma,b}} + \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{X^{0,b}}. \end{aligned}$$

Thereafter, from Lemma 4.3 and Hölder inequality, we obtain

$$\| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \lesssim \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}$$

and

$$\begin{aligned} & \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} = \| |x|^{-2\alpha} * (\langle \nabla \rangle^\sigma (z_1 z_2)) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \\ & \lesssim \|\langle \nabla \rangle^\sigma (z_1 z_2)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^{\frac{d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma (z_1 z_2)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^{\frac{d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \\ & \lesssim (\|\langle \nabla \rangle^\sigma z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} + \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}}) \\ & \times (\|\langle \nabla \rangle^\sigma z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} + \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}}). \end{aligned}$$

Now we use Bernstein inequality and $N_3 \gtrsim N_2$ to get

$$\begin{aligned} & \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \\ & \lesssim N_3^\sigma \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}}. \end{aligned}$$

Therefore, from Lemma 2.4, we conclude that

$$\sum_{N_3 \gtrsim N_2} (4.7) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}}$$

outside a set of probability

$$\leq C \exp \left(-c \frac{R^2}{T^{1-\varepsilon} \|\phi\|_{H^{0+}}} \right).$$

Case (6.ii) : $N_3 \ll N_2 \sim N_1$

We assume $N_4 \geq N_3$, because other case can be handle similarly.

From Hölder inequality and Lemma 3.3, we get

$$(4.7) \lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2) v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\ \lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2) v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{X^{0,1-b-\theta}},$$

for some small positive ε such that $\varepsilon < \frac{1}{\alpha}(\sigma - \frac{\alpha}{2})$. Then Lemma 4.2 and Lemma 3.2 give

$$\|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (|\eta_T v|^2) \eta_T v)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \\ \lesssim \| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|\langle \nabla \rangle^\sigma v_3\|_{L_t^\infty L_x^2} + \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{L_t^\infty L_x^2} \\ \lesssim \| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{X^{\sigma,b}} + \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{X^{0,b}}.$$

Thereafter we use Lemma 4.3 and Hölder inequality to get

$$\| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \lesssim \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}$$

and

$$\|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} = \| |x|^{-2\alpha} * (\langle \nabla \rangle^\sigma (z_1 z_2)) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \\ \lesssim \|\langle \nabla \rangle^\sigma (z_1 z_2)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^{\frac{d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma (z_1 z_2)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^{\frac{d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \\ \lesssim (\|\langle \nabla \rangle^\sigma z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}})^{\frac{1}{2}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} + \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \\ \times (\|\langle \nabla \rangle^\sigma z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} + \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}}).$$

Then by using Bernstein inequality and $N_1 \sim N_2$, we have

$$\|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \lesssim N_1^{\sigma-2s} \prod_{j=1}^2 \|\langle \nabla \rangle^s z_j\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^s z_j\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}}.$$

Therefore, from Lemma 2.4, we conclude that

$$\sum_{N_3 \ll N_2 \sim N_1} (4.7) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}}$$

outside a set of probability

$$\leq C \exp \left(-c \frac{R^2}{T^{1-\varepsilon} \|\phi\|_{H^s}} \right).$$

Case (6.iii) : $N_3, N_1 \ll N_2$

We consider 4 cases separately

- a. $N_4 \sim N_2 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3, N_1$
- b. $N_4 \sim N_2 \gg N_3 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$
- c. $N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3$
- d. $N_4 \sim N_2 \gg N_3, N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}}.$

Subcase (6.iii.a) : $N_4 \sim N_2 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3, N_1$

Similarly to Case (2.ii.a), we need to estimate

$$\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (z_1^k z_2^l v_3) v_4| dx dt.$$

So we consider

$$\begin{aligned} & N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma z_1^k z_2^l v_3 v_4| dx dt, \quad N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt \\ & \text{and } N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma v_3 v_4| dx dt. \end{aligned}$$

Second term will be only considered, because remaining two terms can be handled similarly. By using Hölder inequality, Lemma 3.6 and Lemma 3.7, we have

$$\begin{aligned} & N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt \lesssim N_2^{2\alpha-d} \|\langle \nabla \rangle^\sigma z_2^l v_3\|_{L_{t,x}^2} \|z_1^k v_4\|_{L_{t,x}^2} \\ & \lesssim N_3^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}+2\alpha-d} N_1^{\frac{d-1}{2}+2\epsilon} N_4^{\frac{1-\alpha}{2}+2\epsilon} \|\langle \nabla \rangle^\sigma z_2^l\|_{X^{0,b}} \|v_3\|_{X^{0,b}} \|z_1^k\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\epsilon}} \end{aligned}$$

Thereafter, from $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|\langle \nabla \rangle^\sigma z_2^l\|_{X^{0,b}} = \|\langle \nabla \rangle^\sigma z_2\|_{X^{0,b}}, \|z_1^k\|_{X^{0,b}} = \|z_1\|_{X^{0,b}}$), Bernstein inequality and Lemma 3.1, we obtain

$$\begin{aligned} & \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt \\ & \lesssim N_3^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}+2\alpha-d} N_1^{\frac{d-1}{2}+2\epsilon} N_4^{\frac{1-\alpha}{2}+2\epsilon} \|\langle \nabla \rangle^\sigma z_2\|_{X^{0,b}} \|v_3\|_{X^{0,b}} \|z_1\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\epsilon}} \\ & \lesssim T^{0-} N_3^{\frac{d-1}{2}-\sigma} N_2^{\frac{1-\alpha}{2}+2\alpha-d+\sigma-s} N_1^{\frac{d-1}{2}-s+2\epsilon} N_4^{\frac{1-\alpha}{2}+2\epsilon} \prod_{j=1}^2 \|P_{N_j} \phi^\omega\|_{H^s} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\epsilon}} \end{aligned}$$

Then we carry out summation in N_1 and N_3 so that

$$\begin{aligned} & \sum_{N_3=1}^{N_2^{\frac{\alpha-1}{2\alpha-1}}} \sum_{N_1=1}^{N_2^{\frac{\alpha-1}{2\alpha-1}}} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt \\ & \lesssim T^{0-} N_2^{\alpha \frac{2\alpha-d}{2\alpha-1} + \sigma(\frac{\alpha}{2\alpha-1}) - s(\frac{3\alpha-2}{2\alpha-1}) + 2\epsilon(\frac{3\alpha-2}{2\alpha-1})} \|\phi^\omega\|_{H^s} \|P_{N_3} \phi^\omega\|_{H^s} \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\epsilon}}. \end{aligned}$$

Since $s > \sigma \frac{2\alpha-1}{4\alpha-3} > \sigma \frac{\alpha}{3\alpha-2}$, summation in N_2 can be also carried out.

Hence, from Lemma 2.2, we have

$$\sum_{N_4 \sim N_2 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3, N_1} (4.7) \lesssim T^{0-} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\epsilon}}$$

outside a set of probability

$$C \exp(-c \frac{R^2}{\|\phi\|_{H^s}^2}).$$

Subcase (6.iii.b) : $N_4 \sim N_2 \gg N_3 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$

Similarly to Case (2.ii.a), we need to deal with

$$\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (z_1^k z_2^l v_3) v_4| dx dt.$$

So we have to estimate

$$N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma z_1^k z_2^l v_3 v_4| dx dt, \quad N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt$$

and $N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma v_3 v_4| dx dt.$

Second term will be only considered, because remaining two terms can be handled similarly. Hölder inequality and Lemma 3.7 yield

$$\begin{aligned} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma v_3 v_4| dx dt &\lesssim N_2^{2\alpha-d} \|v_3\|_{L_t^q L_x^r} \|\langle \nabla \rangle^\sigma z_2^l\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|z_1^k v_4\|_{L_{t,x}^2} \\ &\lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_3\|_{L_t^q L_x^r} \|\langle \nabla \rangle^\sigma z_2^l\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|z_1^k\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \end{aligned}$$

Thereafter we use $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|z_1^k\|_{X^{0,b}} = \|z_1\|_{X^{0,b}}$, $\|\langle \nabla \rangle^\sigma z_2^l\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} = \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}}$), Bernstein inequality and Lemma 3.2 to obtain

$$\begin{aligned} &\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma v_3 v_4| dx dt \\ &\lesssim N_3^{\frac{2-\alpha}{q}-\sigma} N_2^{2\alpha-d+\sigma-s} N_1^{\frac{d-1}{2}-s+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|v_3\|_{X^{\sigma,b}} \|z_1\|_{X^{s,b}} \|\langle \nabla \rangle^s z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}, \end{aligned}$$

where $q = \max(3, \frac{3(\alpha-1)(2-\alpha)}{(3\alpha-1)(d-2\alpha)})$.

After that, we carry out summation in N_1 and N_3 and apply Lemma 3.1 as follows

$$\begin{aligned} &\sum_{N_3=N_2^{\frac{\alpha-1}{2\alpha-1}}}^{N_2} \sum_{N_1=1}^{N_2^{\frac{\alpha-1}{2\alpha-1}}} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma v_3 v_4| dx dt \\ &\lesssim N_2^{\frac{\alpha-1}{2\alpha-1}(\frac{2-\alpha}{q} + \frac{d-2\alpha}{2(\alpha-1)}(1-3\alpha)) + \frac{\alpha}{2\alpha-1}\sigma - \frac{3\alpha-2}{2\alpha-1}s + 2\frac{3\alpha-2}{2\alpha-1}\varepsilon} \|v\|_{X^{\sigma,b}} \|z\|_{X^{s,b}} \|\langle \nabla \rangle^s z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \\ &\lesssim T^{0-} N_2^{\frac{\alpha-1}{2\alpha-1}(\frac{2-\alpha}{q} + \frac{d-2\alpha}{2(\alpha-1)}(1-3\alpha)) + \frac{\alpha}{2\alpha-1}\sigma - \frac{3\alpha-2}{2\alpha-1}s + 2\frac{3\alpha-2}{2\alpha-1}\varepsilon} \|v\|_{X^{\sigma,b}} \|\phi^\omega\|_{H^s} \|\langle \nabla \rangle^s z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \end{aligned}$$

Since the exponent of N_2 is negative, summation in N_2 can be also carried out.

Therefore, from Lemma 2.2 and Lemma 2.4, we conclude that

$$\sum_{N_4 \sim N_2 \gg N_3 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1} (4.7) \lesssim T^{0-} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}$$

outside a set of probability

$$C \exp(-c \frac{R^2}{\|\phi\|_{H^s}^2}) + C \exp(-c \frac{R^2}{T^{\frac{q-2}{q}} \|\phi\|_{H^s}^2}).$$

Case(6.iii.c) : $N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3$

Similarly to Case (2.ii.a), we estimate the following :

$$\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma (z_1^k z_2^l v_3) v_4| dx dt.$$

So we need to handle

$$N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |\langle \nabla \rangle^\sigma z_1^k z_2^l v_3 v_4| dx dt, \quad N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt$$

and $N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k z_2^l \langle \nabla \rangle^\sigma v_3 v_4| dx dt.$

Second term will be only considered, because remaining two terms can be handled similarly. Hölder inequality and Lemma 3.7 give

$$\begin{aligned} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt &\lesssim N_2^{2\alpha-d} \|z_1^k\|_{L_{t,x}^4} \|\langle \nabla \rangle^\sigma z_2^l\|_{L_{t,x}^4} \|v_3 v_4\|_{L_{t,x}^2} \\ &\lesssim N_2^{2\alpha-d} N_3^{\frac{d-1}{2}+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \|z_1^k\|_{L_{t,x}^4} \|\langle \nabla \rangle^\sigma z_2^l\|_{L_{t,x}^4} \|v_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}} \end{aligned}$$

Then, from $\sum_{k,l} |C_{k,l}| \leq C$ (with $\|z_1^k\|_{L_{t,x}^4} = \|z_1\|_{L_{t,x}^4}$, $\|\langle \nabla \rangle^\sigma z_2^l\|_{L_{t,x}^4} = \|\langle \nabla \rangle^\sigma z_2\|_{L_{t,x}^4}$) and Bernstein inequality, we obtain

$$\begin{aligned} &\sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt \\ &\lesssim N_1^{-s} N_2^{2\alpha-d+\sigma-s} N_3^{\frac{d-1}{2}-\sigma+2\varepsilon} N_4^{\frac{1-\alpha}{2}+2\varepsilon} \prod_{j=1}^2 \|\langle \nabla \rangle^s z_j\|_{L_{t,x}^4} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}. \end{aligned}$$

Since $N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3$, $\frac{d-1}{2} - \sigma + 2\varepsilon > 0$ and $-s < 0$, we can carry out summation in N_1 and N_3 such that

$$\begin{aligned} &\sum_{N_1=N_2^{\frac{\alpha-1}{2\alpha-1}}}^{N_2} \sum_{N_2=1}^{N_2^{\frac{\alpha-1}{2\alpha-1}}} \sum_{k,l \in \mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int \int_{\mathbb{R} \times \mathbb{R}^d} |z_1^k \langle \nabla \rangle^\sigma z_2^l v_3 v_4| dx dt \\ &\lesssim N_2^{\frac{(d-2\alpha)(1-3\alpha)}{2(2\alpha-1)} + \frac{\alpha}{2\alpha-1}\sigma - \frac{3\alpha-2}{2\alpha-1}s + \frac{3\alpha-2}{2\alpha-1}\varepsilon_1} \|\langle \nabla \rangle^s z\|_{L_{t,x}^4} \|\langle \nabla \rangle^s z_2\|_{L_{t,x}^4} \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \end{aligned}$$

After carrying out summation in N_2 , from Lemma 2.4, we conclude that

$$\sum_{N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3} (4.7) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}$$

outside a set of probability

$$C \exp(-c \frac{R^2}{T^{\frac{1}{2}} \|\phi\|_{H^s}^2}).$$

Subcase(6.iii.d) : $N_4 \sim N_2 \gg N_3, N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}}$

Hölder inequality and Lemma 3.3 yield (4.7) is bounded by

$$\|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2) v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2) v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{X^{0,1-b-\theta}},$$

for some small ε such that $0 < \varepsilon < \frac{1}{\alpha}(\sigma - \frac{\alpha}{2})$. Then we use Lemma 4.2 and Lemma 3.2 to obtain

$$\begin{aligned} & \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2) v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \\ & \lesssim \| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|\langle \nabla \rangle^\sigma v_3\|_{L_t^\infty L_x^2} + \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{L_t^\infty L_x^2} \\ & \lesssim \| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{X^{\sigma,b}} + \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{X^{0,b}}. \end{aligned}$$

Thereafter, from Lemma 4.3 and Hölder inequality, we obtain

$$\begin{aligned} & \| |x|^{-2\alpha} * (z_1 z_2) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \lesssim \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \\ & \lesssim N_1^{-s} N_2^{-s} \|\langle \nabla \rangle^s z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|\langle \nabla \rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \end{aligned}$$

and

$$\begin{aligned} & \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} = \| |x|^{-2\alpha} * (\langle \nabla \rangle^\sigma (z_1 z_2)) \|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \\ & \lesssim \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^{\frac{d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^{\frac{d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \\ & \lesssim (\|\langle \nabla \rangle^\sigma z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} + \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}}) \\ & \times (\|\langle \nabla \rangle^\sigma z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} + \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}}). \end{aligned}$$

Then we use Bernstein inequality and $N_2 \gtrsim N_1$ to get

$$\|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \lesssim N_2^{\sigma-s} N_1^{-s} \prod_{j=1}^2 \|\langle \nabla \rangle^s z_j\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^s z_j\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}}.$$

Thus, from $N_2 \gtrsim N_3$, we have

$$\begin{aligned} & \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2) v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \\ & \lesssim N_2^{\sigma-s} N_1^{-s} N_3^{-\sigma} \|\langle \nabla \rangle^s z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|\langle \nabla \rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|v_3\|_{X^{\sigma,b}} \\ & + N_2^{\sigma-s} N_1^{-s} N_3^{-\sigma} (\prod_{j=1}^2 \|\langle \nabla \rangle^s z_j\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^s z_j\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}}) \|v_3\|_{X^{\sigma,b}}. \end{aligned}$$

Now we carry out summation in N_1 and N_3

$$\begin{aligned}
& \sum_{N_3=N_2^{\frac{\alpha-1}{2\alpha-1}}}^{N_2} \sum_{N_1=N_2^{\frac{\alpha-1}{2\alpha-1}}}^{N_2} \|\langle \nabla \rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2) v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \\
& \lesssim N_2^{\frac{\alpha}{2\alpha-1}\sigma - \frac{3\alpha-2}{2\alpha-1}s} \left(\|\langle \nabla \rangle^s z\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|\langle \nabla \rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v\|_{X^{\sigma,b}} + \right. \\
& \left. \|\langle \nabla \rangle^s z\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^s z\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}^{\frac{1}{2}} \|\langle \nabla \rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}^{\frac{1}{2}} \|v\|_{X^{\sigma,b}} \right)
\end{aligned}$$

Therefore, from Lemma 2.4, we conclude that

$$\sum_{N_4 \sim N_2 \gg N_3, N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}}} (4.7) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,1-b-\theta}}$$

outside a set of probability

$$\leq C \exp \left(-c \frac{R^2}{T^{1-\varepsilon} \|\phi\|_{H^s}} \right).$$

□

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REFERENCES

- [1] T. Alazard, R. Carles, Loss of regularity for supercritical nonlinear Schrödinger equations, *Math. Ann.* **343** (2009), no. 2, 397420.
- [2] J. Bergh, J. Lofstrom, *Interpolation spaces*, Springer-Verlag, 1976, No. 223.
- [3] Á. Bényi, T. Oh And O. Pocovnicu, Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS, to appear in *Excursions in Harmonic Analysis*.
- [4] Á. Bényi, T. Oh And O. Pocovnicu, On the probabilistic Cauchy theory of the cubic nonlinear Schrodinger equation on \mathbb{R}^d , $d \geq 3$. (arXiv:1405.7327)
- [5] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity*, *Internat. Math. Res. Notices* 1998, no. 5, 253-283S.
- [6] J. Bourgain, Invariant measures for the Gross-Piatevskii equation, *J. Math. Pures Appl.* (9) **76** (1997), no. 8, 649702
- [7] J. Bourgain, A. Bulut, Almost sure global well posedness for the radial nonlinear Schrodinger equation on the unit ball II: the 3D case, to appear in *J. Eur. Math. Soc*
- [8] N. Burq, P. Gérard, N. Tzvetkov, An instability property of the nonlinear Schrödinger equation on \mathbb{S}^d , *Math. Res. Lett.* **9**(23), 323335 (2002)
- [9] ———, Two singular dynamics of the nonlinear Schrödinger equation on a plane domain. *Geom. Funct. Anal.* **13**(1), 119 (2003)
- [10] N. Burq, N. Tzvetkov, Random data Cauchy theory for supercritical wave equations. I. Local theory, *Invent. Math.* **173** (2008), no. 3, 449475.

- [11] N. Burq, N. Tzvetkov, Probabilistic well-posedness for the cubic wave equation, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 1, 130.
- [12] R. Carles, Geometric optics and instability for semi-classical Schrodinger equations, Arch. Ration. Mech. Anal. **183** (2007), no. 3, 525-553.
- [13] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics **10**, American Mathematical Society. (2000)
- [14] M. Chae, Y. Cho, S. Lee, *Mixed norm estimates of Schrodinger waves and their applications*, Comm. Partial Differential Equations **35** (2010), 906-943.
- [15] Y. Cho, G. Hwang and C. Yang, *Remarks on the mass-critical fractional Hartree equations*, in preparation.
- [16] Y. Cho, H. Hajaiej, G. Hwang and T. Ozawa, *On the Cauchy problem of fractional Schrödinger equation with Hartree type nonlinearity*, Funkcialaj Ekvacioj **56** (2013) 193-224
- [17] Y. Cho, G. Hwang, S. Kwon and S. Lee, *Profile decompositions and blowup phenomena of mass critical fractional Schrödinger equations*, Nonlinear Analysis **86** (2013), 12-29.
- [18] Y. Cho and S. Lee, *Strichartz estimates in spherical coordinates*, Indiana Univ. Math. J. **62** (2013), no. 3, 991-1020.
- [19] Y. CHO AND T. OZAWA, *On the semi-relativistic Hartree type equation*, SIAM J. Math. Anal., **38** (2006), No. 4, 1060–1074.
- [20] Y. Cho, G. Hwang and Y. Shim, *Energy concentration of the focusing energy-critical FNLS*, (arXiv:1502.00100)
- [21] Y. Cho, T. Ozawa, S. Xia, *Remarks on some dispersive estimates*, Commun. Pure Appl. Anal., **10** (2011), no. 4, 1121-1128.
- [22] M. Christ, J. Colliander, T. Tao, Asymptotics, frequency modulation, and low-regularity illposedness of canonical defocusing equations, Amer. J. Math. **125** (2003), no. 6, 1235-1293.
- [23] J. Colliander, T. Oh, Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$, Duke Math. J. **161** (2012), no. 3, 367-414.
- [24] Y. Deng, Two-dimensional nonlinear Schrodinger equation with random radial data, Anal. PDE **5**(2012), no. 5, 913-960.
- [25] Z. Guo, Y. Sire, Y. Wang and L. Zhao, *On the energy-critical fractional Schodinger equation in the radial case*, (arXiv:1310.6816)
- [26] Z. Guo and Y. Wang, Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations, in preprint. (arXiv:1007.4299v3)
- [27] N. Hayashi and T. Ozawa, *Smoothing effect for Schrödinger equations*, J. Fuctional. Anal. **85** (1989), 307-348.
- [28] S. Herr and E. Lenzmann, *The Boson star equation with initial data of low regularity*, Nonlinear Anal. **97** (2014), 125-137.
- [29] Y. Hong, Y. Sire, *On Fractional Schrodinger Equations in sobolev spaces*, (arXiv:1501.01414)
- [30] T. KATO, *On nonlinear Schrödinger equations II. H^s -solutions and unconditional well-posedness*, J. Anal. Math., **67** (1995), 281–306.
- [31] S. Keraani, A. Vargas, *A smoothing property for the L^2 -critical NLS equations and an application to blowup theory*, Ann. Inst. H. Poincare Anal. Non Lineaire **26** (2009), no. 3, 745–762.
- [32] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A, **268** (2000), 298-305.
- [33] ———, *Fractals and quantum mechanics*, Chaos, **10** (2000), 780-790.
- [34] ———, *Fractional Schrödinger equation*, Phys. Rev. E, (3) **66** (2002), no. 5, 056108, 7 pp.
- [35] J. Lührmann and D. Mendelson *Random Data Cauchy Theory for Nonlinear Wave Equations of Power-Type on \mathbb{R}^3* , Comm. Partial Differential Equations **39:12** (2014), 2262-2283

- [36] M. Chae, Y. Cho, S. Lee, *Mixed norm estimates of Schrodinger waves and their applications*, Comm. Partial Differential Equations **35** (2010), 906-943.
- [37] C. Miao, G. Xu and L. Zhao, *Global well-posedness and scattering for the mass-critical Hartree equation with radial data*, J. Math. Pure Appl. **91** (2009), 49-79.
- [38] ———, *Global well-posedness and scattering for the energy-critical, defocusing Hartree equation for radial data*, J. Func. Anal. **253** (2007), 605-627.
- [39] ———, *Global Well-Posedness and Scattering for the Energy-Critical, Defocusing Hartree Equation in \mathbb{R}^{1+n}* , Commun. Partial Differential Equations **36** (2011), 729-776.
- [40] A. Nahmod, N. Pavlovic and G. Staffilani, *Almost sure existence of global weak solutions for super-critical Navier-Stokes equations*, SIAM J. Math. Anal. **45** (2013), 34313452
- [41] K. Nakanishi, *Modified wave operators for the Hartree equation with data, image and convergence in the same space. II*, Ann. Henri Poincaré. **3** (2002), 503 - 535.
- [42] T. Oh and O. Pocovnicu, *Probabilistic global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on \mathbb{R}^3*
- [43] O. Pocovnicu, *Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on \mathbb{R}^d , $d = 4$ and 5*
- [44] L. Thomann, *Random data Cauchy problem for supercritical Schrodinger equations*, Ann. Inst. H. Poincaré Anal. Non Lineaire **26** (2009), no. 6, 23852402.
- [45] Y. Tsutsumi, *L^2 -solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. **30** (1987), 115-125.

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